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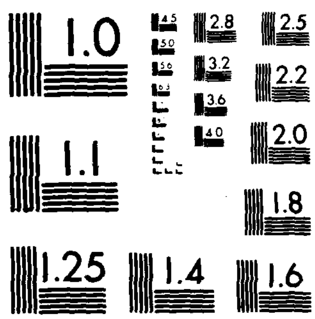

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CLIFFORD MARSHALL

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ABSTRACT

From a customers point of view, warranties seem desirable as a means of improving reliability (availability)and reducing overall system cost. However contractors may be reluctant to provide warranties, particularly when an assumption of excessive risk is perceived to exist. In such situations a mutually agreeable arrangement may be achieved by means of an incentive fee type of warranty. This point of view is similar to the various incentive contract forms introduced some twenty years ago to divide risk assumption between the customer and the contractor. In incentive contracts it is usual to have some linear decrease in fee as cost increases. To carry the incentive concept into the structuring of warranties a different point of view is required, based on the same general concepts of dividing risk assumption. Incentivising reliability warranties is achieved by providing a warranty fee pool as a function of time. Use of an increasing fee pool provides the incentive for a contractor to assume the risk involved in a warranty situation by sharing risk with the customer. It is also possible for the contractor to achieve additional profit under properly structured incentivised warranties.

The research reported here defies the concepts of incentivised reliability (availability) warranties, develops mathematical models of the structure of such warranties, and analyzes selected examples. The goal of the analyses is to identify and study the role of contract variables as they relate to the division of risk assumption, the improvement of reliability (availability) and the control of cost in a warranty situation.

Since the research deals with reliability considerations it is carried out in a stochastic framework, where the mathematical models provide expected values and standard deviations of costs for repair/replacement as functions of time. Warranty fee pools are represented by various functional forms and different probability laws are employed to govern the failure events and times to failure.

Incentivising Availability Warranties

CONTENTS

Chapter 1. Basic Concepts and Methodology.

From the point of view of the user, delivery of a piece of equipment is only the beginning of a period over which the equipment is required to function, its lifetime. On the other hand a contractor or supplier may view the delivery as the end of its period of concern with the equipment. These two viewpoints do not represent desirable approaches to equipment acquisition and in actual practice the user and supplier broaden their periods of interest in the equipment, both being involved throughout at least part of the development and the use periods. User involvement in development occurs through design specifications, design reviews, and various acceptance test procedures. Supplier involvement in equipment operation occurs in some form of maintenance agreement to be found in most contracts for equipment acqusition. One way to achieve satisfactory use is by improving the reliability of equipment. However reliability is a difficult quantity to design, being composed of various random features. A supplier often agrees to some level of reliability as part of a contract but in fact the actual level of reliability may be found to differ from specified values once the equipment is operational. In addition, being random, the times to failure of equipment may prove to be considerably short while still falling technically within design specifications.

In recent years there has been an attempt to improve reliability by having the contractor assume the risk of an equipment failure in terms of a warranty on the equipment. The Reliability Improvement Warranty (RIW) concept has received considerable attention and analysis, as typified by the studies reported in (1). One of the main features of an RIW is that it forces a contractor to take a serious view regarding reliability. This is not done in many cases because of the difficulty associated with quantifying reliability

in a significant way. Data on reliability are apt to be scarce or inaccurate. The concepts, based on probability theory, seem strange to most management and engineering personnel, and therefore only cursory attention may be paid to contractual specifications in the reliability area. The RIW makes reliability considerations important from a monetary point of view, acting as a form of penalty against profits should the equipment need to be repaired or replaced during the warranty period.

From the users point of view the RIW has two important features. It focuses the contractor on reliability considerations which are likely to improve the operational status of equipment, and requires the contractor to share the risk of failing to achieve a satisfactory level of operation.

The RIW concept introduces two possibilities for generalization. Though it is desirable from the users point of view, contractors may be very reluctant to assume the full level of risk implied in a normal warranty contract. This leads to the consideration of various incentive forms of warranties in which the risk can be shared between the contractor and the user. Furthermore in such situations there can exist the possibility of additional profit for the contractor when high reliability is achieved. Since such achievement is often the major goal of the user as well, he has a strong motivational instrument in the properly structured incentive warranty. The other direction of generalization is to the concept of availability which, in fact, is most often the users concern rather than purely reliability alone. Equipment failure can often be tolerated so long as down time is sufficiently short and the time between failures satisfactorily long.

This report studies the concept of incentivisation of warranties for the reliability and more generally, availability situations. The methodology

is based on mathematical models of warranty contract structures in a stochastic framework as described in this chapter.

Section 1.1 Introduction

Incentive type contracts have been in use for some time. They are designed to motivate a contractor to keep cost low, meet schedules, or achieve high levels of performance. The most common form of incentive contracts provide for a fee that decreases with increasing cost. In some cases the decrease may reach to a negative fee (penalty), whereas other forms of contract limit the decrease. Most often the decrease in fee is linear or consists of a few linear segments differing in their rate of decrease. Incentive contracts, in their various forms, have been studied extensively and widely applied. Theoretical analysis of such contract forms may be found in (2) and (3).

The incentive fee concept was developed as an instrument for causing the contractor to assume part of the risk in situations where cost was difficult to determine. Incentivised contracts fall between fixed fee situations where costs should be deterministic and purely research type areas where cost plus fixed fee contracts are required before a contractor will engage in the necessary work. Thus it is the relative uncertainty of cost that makes an incentive fee contract desirable to both contractor and customer. Such concepts may be applied to the reliability/availability situation. If the availability profile of a piece of equipment was deterministic a contractor could figure a meaningful addition to cost for including a warranty. At the other extreme, if the availability behavior was completely erratic no contractor would be willing to provide a warranty. Thus the RIW situation can only be agreed to when a contractor believes he can correctly estimate the availability profile and include sufficient cost to compensate his assumption

of the warranty. This situation may not be expected to occur too often, particularly if the contractor fully understands the random character of availability. However the use of incentives described above suggests a method for greatly enlarging the range of situations in which a contractor may engage in a warranty. Rather than having only the two extreme situations of: no warranty, with user assuming all the risk, and normal warranty, with contractor assuming all the risk, the incentivised warranty allows a distribution of risk assumption between user and contractor.

To carry over the incentive concept into the structuring of warranties a different point of view is required, though still following the same general principle of dividing risk assumption. Incentivising reliability warranties is achieved by providing a warranty fee pool as a function of time. Use of an increasing fee pool provides the incentive for a contractor to assume the risk involved in a warranty situation by sharing risk with the customer. An increasing fee pool motivates the contractor to increase time between failures as much as possible which is one major feature of high availability. It is also possible for the contractor to achieve additional profit under a properly structured incentivised warranty. However such situations must be treated very carefully to prevent motivating the contractor to allow failure, which is counter to a major feature of high availability i.e. few failures. A major reason for the detailed mathematical analysis of incentivised availability warranties is to clearly define the various features implicit in such relatively complex contract structures and to quantify, within postulated stochastic frameworks, the levels and forms of incentivisation that will provide desirable availability profiles at a cost agreeable to both the contractor and the customer.

Section 1.2 Definitions and Terminology

This Section gives the major definitions required for the formulation of mathematical models in the study of incentivised availability warranties.

The models will consider the payment by the contractor under its warranty obligation as a stochastic process taking place over the warranty period. The models considered in this report will deal with situations in which a failure causes a payment "c" to be made by the contractor at the time of failure. This may represent replacement or repair costs. The time for making repairs or replacements is not considered in these models. It can be assumed to be relatively short with respect to the full warranty period. The present models are intended to give some basic insights into the incentive warranty concept and it seems undesirable to include the complications of repair/replacement times in these initial models. Further research may well include those times, providing additional insights into the picture of the availability profile.

In a stochastic availability model the number of times an equipment fails in the warranty period is a discrete random variable. A general model can be made with this point of view and one form of such a model is discussed in Chapter 4. However for the purpose of gaining insight into the incentive warranty concept such general models are far too complex. The mathematical and stochastic features of the general model tend to overshadow the details of particular interest in studying the incentive concept. Therefore the major work of this report, described in Chapters 2 and 3 employs simplified models in which at most one or at most two failures can occur within the warranty period. Technically these models may be thought of as conditional models formulated under the conditional probabilities for the

number of failures. Since the models are self contained there is no need to include the actual conditioning features in the details of these simplified models. The one or two failure type models contain the concepts necessary to study incentivised warranties and in many practical situations (though certainly not all) at most two failures may be expected in a reasonable warranty period.

The case of at most one failure in the warranty period is the reliability case. Thus its study, in Chapter 2, constitutes an extension of the RIW to incentivised forms.

Some of the basic notation used in these studies will now be given, additional notation will be provided as it is required.

The warranty period is denoted by T^*. It is a contract parameter which specifies the duration of the contractors obligation for assuming correction of failures under the warranty.

The warranty fee pool, denoted by $W(t)$ is an amount of money that the customer will provide toward the repair replacement cost for a failure at time t. The fee pool forms employed in this study are of linear segment type. They can most simply be described in the one failure (reliability) case.

The linear segment form in the one failure case is:

$$\begin{aligned} W_1(t) &= a_o \qquad , \quad 0 \leq t \leq t' \\ &= a_o + b(t-t'), \quad t' \leq t \leq T^* \end{aligned}$$

where a constant pool value a_o is provided for part of the warranty period, specified by t', and the remaining amount increases linearly at a rate measured by b. This type of warranty fee pool is specified by three parameters (contract parameters): a_o, b, t' and represents a reasonable

form for such a pool. It is selected because of its desirability as an actual form rather than for any mathematical convenience. Therefore it acts as a valuable model form in studying the concepts of incentivised warranties. Other warranty pool forms might be employed, for example one based on exponential type functions. However it is felt that the linear segment type is best suited for contract negotiations and for sound intuitive understanding of the features implicit in incentive warranty structures. The purpose of this study is to contribute to such understanding and this was the reason for selecting the linear segment form of warranty fee pool. Some calculations are cumbersome with this form but it is not clear that other forms would greatly simplify calculation. Of course there is no reason to limit the warranty fee pool to linear segments and any monotonic non-decreasing function could be used.

For more than one failure the fee pool $W(t)$ becomes a stochastic process. Each time there is a failure some kind of adjustment must be made in $W(t)$ to account for this occurrence. The treatment employed in this study is as follows:

Let $w_k(t)$ denote the warranty fee pool function after k failures. In this notation $w_o(t) = W_1(t)$ defined above. If the first failure occurs at t_1 then

$$
\begin{aligned}
 &= 0 &&, \ t < t_1 \\
w_1(t) &= w_o(t_1) - c &&, \ t_1 \leq t \leq t' + t_1 \\
 &= w_o(t_1) - c + b(t-t_1-t'), && t'+t_1 \leq t \leq T^*
\end{aligned}
$$

Should t' exceed the remaining time T^*-t_1 then $w_1(t)$ has the constant value $w_o(t_1) - c$ over the remaining period.

If the second failure occurs at time t_2 then:

$$
\begin{aligned}
w_2(t) &= 0 &&, \ t < t_2 \\
&= w_1(t_2) - c &&, \ t_2 \le t \le t' + t_2 \\
&= w_1(t_2) - c + b(t-t_2-t'), && \ t' + t_2 \le t \le T^*.
\end{aligned}
$$

In general a recurressive definition specifies $w_k(t)$ in terms of $w_{k-1}(t_k)$ where t_k is the time of the k^{th} failure. In turn $W(t)$ is defined in terms of these $w_k(t)$ values and the joint failure time distribution over the warranty period. This general formulation will be treated in Chapter 4.

The random nature of the failure events is determined by the failure time distributions and the number of failures. When there is at most one failure the failure time defines the random characteristics of the model. These studies make use of two kinds of failure time distributions: uniform and exponential, for detailed analysis. General formulations are given in terms of general failure distributions. The exponential distribution of the form:

$$P[T \le t] = 1 - e^{-\lambda t}$$

is widely used in reliability work and has been found to represent many actual situations. Its relative simplicity and wide applicability make it a good type of distribution to use for some of the present studies. In addition many people interested in reliability and related fields, such as the one under study here, will be familiar with exponential type models.

The distribution used most in the one failure case (Chapter 2) is not the exponential but the uniform distribution having the simple form:

$$
\begin{aligned}
f_T(t) &= \frac{1}{s-r}, && r \le t \le s \\
&= 0 , && \text{elsewhere}
\end{aligned}
$$

for the probability density function for the failure time T. Calculations

using the uniform density are rather complicated, though the exponential does not greatly simplify calculations as shown e.g. in Chapter 2. It may be felt that the uniform distribution does not represent many actual situations of random failure. However the reason for using the uniform distribution as a major feature of the mathematical models for incentivised warranties is that their use allows detailed study of the major features of incentive warranties. Moreover actual failure situations can be both approximated and bounded by the uniform distribution type models. Their use directs attention to the features of the warranty sturcture rather than overshadowing them in the mathematics of more involved random processes.

The models deal with the stochastic process $A_k(t)$, the payment by the constractor under its warranty obligations when k failures occur. The process is denoted by $A(t)$ in the unconditional case where the number of failures is unspecified, as discussed in Chapter 4. Chapter 2 deals with $A_1(t)$, the reliability case and Chapter 3 deals with $A_2(t)$, the simplest case that is of availability type. Analysis of the models considers the expected values $E[A_k(t)]$ and variances $Var[A_k(t)]$ which are taken as the quantities which represent the significant features of incentive warranties. By considering the contract parameters present in the warranty fee pool, the parameters of the underlying failure time distributions, and the form of the process $A_k(t)$ an understanding of the complex warranty structure is obtained.

Section 1.3 Scope of Model Methodology

The mathematical models employed in this study are of three general types: at most one failure, at most two failures, and the general case of unspecified number of failures. Each model employes a specified form of warranty incentive fee pool, and probability distribution of time between failures.

The study is limited to a single piece of equipment and its availability profile over a specified warranty period. Correction of failure by replacement or repair is assumed to have fixed cost and to be instantaneous. All of these conditions can, of course, be generalized and it may be of value to do so in subsequent studies. In this initial investigation the objective is not to treat the most general or even the most realistic situations but rather to study the concept of incentivising warranties and consider their generalization from reliability to availability form.

Each model provides an opportunity for analysis designed to show the effect on the payment process $A_k(t)$ due to postulated structure and magnitude of contract parameters defining the incentive fee pool. These effects must necessarily be studied within postualted random behavior of equipment failure. The results are intended to show how incentivising warranties motivates a contractor to both assume such warranty obligations and provide a desirable level of availability. The analyses of this study constitute a background for understanding the incentivising of availability warranties. Though the study does not intend to provide a detailed plan for structuring actual warranties it should serve as an initial guide toward applications. The incentive fee pool form used in many of the models, a linear segment type, seems reasonable for actual contract structuring and negotiation. This form is based on a structure that can be intuitively appreciated by contract negotiators with a wide range of technical background. Moreover they are based on a few parameters whose effects can be directly related to both contractor and customer goals through analyses similar to those contained in this study.

An important consideration in warranties is the value of money. This is because the payment process $A_k(t)$ will take place in the future, often several years after the original contract for development and manufacture of

an equipment is negotiated. It has become rather standard practice in studying contract structuring and particularly in RIW research, e.g. in (1), to account for the value of money. The present study is primarily intended as a theoretical investigation of the concepts discussed above. The modifications to true value of money do not effect these theoretical considerations, though they must be considered when using the model analyses as guides for structuring actual incentivised warranties. Therefore most of the present study is carried out without regard to value of money considerations. However to provide some indications of how value of money considerations may be included in the analyses some of the work reported here does include them. Though the treatment of present value of future expenditure is "well known" in certain technical areas the basic ideas, as employed in the present study, are included here for completeness.

If an expenditure of amount s is to be made at time t the present value that may be assigned to that expenditure is determined by the discount rate r. This rate is a dimensionless quantity which specifies how the value of money decreases over time due to a combination of factors.

At time t the expenditure s is made. Its value one time unit (e.g. a year) earlier is $s-rs$ or $(1-r)s$. Similarly if the value k time units earlier is denoted by v_k then $v_{k-1} = (1-r)v_k$ so that at the present time $(t=0)$ the value is $(1-r)^t s$. Some work dealing with present value considerations employes an exponential form for the discount rate (e.g. in (1)). In this formulation the "rate" is most often defined as a quantity γ having the dimension of inverse time. The reduction factor is written as an exponential $e^{-\gamma}$ so that present value of an expenditure of amount s at time t becomes: $s\,e^{-\gamma t}$.

More than one future expenditure can be combined into a present value calculation as illustrated by the following case of two such expenditures c_1 and

c_2 taking place at times t_1 and t_2 respectively. Note that in the models considered here the discount rate is assumed to be constant. In the kind of applications under consideration in this study the expenditures c_1 and c_2 may represent the actual cost to the contractor under its warranty oblications i.e. $c-W(t_1)$ and $c-W(t_2)$. Here (t_1, t_2) is a realization of the bivariate random variable (T_1, T_2). The present value of such expenditures (a random variable) is

$$v = c_1(1-r)^{t_1} + c_2(1-r)^{t_2}$$

$$= c\left[(1-r)^{t_1} + (1-r)^{t_2}\right] - W(t_1)(1-r)^{t_1} - W(t_2)(1-r)^{t_2} .$$

In this expression $c\left[(1-r)^{t_1} + (1-r)^{t_2}\right]$ gives the present value of expenses when there is no incentive fee pool. $W(t_1)(1-r)^{t_1} + W(t_2)(1-r)^{t_2}$ gives the present value of the amount contributed by the fee pool.

Though most actual situations may be expected to involve only a few failures with their associated costs, it is interesting to consider a special case that allows any number of failures to occur. Such cases are widely applied as relatively simple mathematical approximations to lifetime availability profiles and indicate bounding values of equipment behavior. To produce a useful model of this kind it is helpful to assume that the failures occur at regular time intervals of duration τ. This is the kind of model obtained from so called expected value techniques where τ is the mean time between failures (e.g. (1) employes such an approach in some of its development). In such a model the present value of all expenditures is:

$$v = c \sum_{k=0}^{\infty} (1-r)^{k\tau} - \sum_{k=0}^{\infty} W(k\tau)(1-r)^{k\tau}$$

where it is assumed that the first failure occurs at time zero. The present value of expenditures when there is no incentive pool, represented by the first infinite sum above, is equal to:

$$\frac{c}{1 - (1-r)^{\tau}} ,$$

a common form of expression for the present value of a series of equal expenditures projected infinitely far into the future. The second expression is in general complicated by the presence of the stochastic process $W(t)$. However it represents the present value of the incentive fee pool process. A bounding special case, which provides at least some insight into the full present value quantity v may be obtained by considering the situation in which the incentive pool is constant and equal to a value d. All this means is that the cost is reduced from c to $c-d$ at each expenditure and

$$v = \frac{c-d}{1 - (1-r)^{\tau}} .$$

A more interesting special case is obtained by assuming that the fee pool values increase to approach d. An appropriate form is $W(k\tau) = d(1-e^{-k\tau}$ where $d \leq c$. Then

$$v = \frac{c-d}{1 - (1-r)^{\tau}} + \frac{d}{1 - e^{-\tau}(1-r)^{\tau}} ,$$

which is greater than in the simple constant fee pool case, as it should be, representing the present value of contractors expenditure under the warranty obligation. The exponentially increasing fee pool provides a smaller contribution from the customer.

Chapter 2 At Most One Failure (The Reliability Case)

This Chapter defines and analyzes models representing incentivised reliability warranties. At most one failure is allowed in these models. The first model, given in Section 2.1, is very simple. It is a deterministic failure time model, included to provide a limit case for the other models. Section 2.2 uses the linear segment type fee pool with three cases of uniform failure time distributions: in case 1 the failure must occur during the period of constant pool value a_o, in case 2 it may occur during constant value a_o or while the fee pool is increasing at constant rate b, the third case deals with failure occurring only within the increasing pool time interval. A fourth case could be considered, allowing the failure to occur toward the end of the increasing pool period or after the warranty period, specified by time T^*, has ended. This case does not seem to contribute basic information of value not already given by the other cases and is therefore not included as part of these studies.

The uniform distributions of Section 2.2 are used to provide detailed structural analysis of how the fee pool operates as a motivational instrument. They are used as tools of analysis rather than selected as representing forms of greatest analytic convenience or representational accuracy. The later points of view often call for use of exponential type failure distribution and these are considered in Section 2.3. The respective utility of the two types of distributions is indicated within their individual sections of analysis. It may be noted that the uniform type failure distribution allows an interesting limit process to be formulated, converging to different values depending on the point of view one assumes. One possibility leading to the deterministic case of Section 2.1 as expected. However the existence of different limit results represents a mathematical formulation of certain qualitative considerations. This is an instance of the trend in some recent applications of mathematics away from

purely quantitative results, as for example in the philosophy and application of catestrophy theory.

Section 2.4 discusses the findings implicit in the analyses of the other sections. It also includes some considerations dealing with the present value of money. For simplicity the basic models do not include the concept of present value since they are intended as theoretical studies rather than models of actual warranty structures.

All sections use the reliability case fee pool of the form:

$$
\begin{aligned}
W(t) &= a_o \qquad , \quad 0 \leq t < t' \\
&= a_o + b(t-t'), \quad t' \leq t \leq T^* .
\end{aligned}
$$

The analyses of this chapter develop expressions for $E[A_1(t)]$ and $Var[A_1(t)]$, the expected value and variance of the contractor payment process $A_1(t)$ in the case of at most one failure. Expected values are computed using condition type logic of the form: For a random variable X defined over an interval (O, R) with distribution function $F_X(x)$

$$
E[X] = \int_0^R x \, d\, F_X(x) = \int_0^{R_1} x \, d\, F_X + \int_{R_1}^{R} x \, d\, F_X
$$

where $0 < R_1 < R$. Thus in dealing with the linear segment fee pool differing payment forms are divided into appropriate segments for computing expected values. The variance formulas make use of the form:

$$
Var[X] = E[X^2] - \left(E[x]\right)^2
$$

properly divided to account for segnented parts of the random variable definition as detailed within the appropriate sections.

These basic ideas can be used to formulate general formulas for $E[A_1(t)]$ and $Var[A_1(t)]$ to be used in the analyses of this chapter.

Here $A_1(t) = 0 \quad , \; t < T$

$$= c - W(T), \quad T \le t \le T^*$$

where the random variable T is the failure time governed by a probability density function $f_T(t)$.

In the following development of basic formulas $P[\;\;]$ denotes the probability expression and a vertical line, as usual, indicates conditional expressions.

$$E[A_1(t)] = E[A_1(t) | t < T] \; P[t < T] + E[A_1(t) | t \ge T] \; P[t \ge T] \; .$$

Since $E[A_1(t) | t < T] = 0$ it follows that:

$E[A_1(t)] = E[c - W(t) | t \ge T] \; P[t \ge T]$, or using the density function $f_T(t)$:

$$E[A_1(t)] = \int_0^t [c - W(v)] f_T(v | v \le t) dv \int_0^t f_T(v) dv$$

However the conditional density is: $f_T(v | v \le t) = \dfrac{f_T(v)}{\int_0^t f_T(v) dv}$

So that:

$$E[A_1(t)] = \int_0^t [c - W(v)] f_T(v) dv.$$

Considering the expression for variance gives:

$$\mathrm{Var}[A_1(t)] = E\left[(A_1(t) - E[A_1(t)])^2\right] ,$$

$$\mathrm{Var}[A_1(t)] = \int_0^t \left(c - W(v) - E[A_1(t)]\right)^2 f_T(v) dv ,$$

Alternatively, using:

$$\mathrm{Var}[A_1(t)] = E\left[\left(A_1(t)\right)^2\right] - \left(E[A_1(t)]\right)^2$$

gives the formula:

$$\mathrm{Var}[A_1(t)] = \int_0^t \left(c - W(v)\right)^2 f_T(v)dv - \left(E[A_1(t)]\right)^2 .$$

These formulas for expected value and variance are employed in the analyses of this chapter.

Section 2.1 Deterministic Failure

When the failure occurs at a specified time s with probability one the deterministic failure model results. It is the simplest case and provides an extreme limit for the other models, particularly the uniform case.

Model analysis derives the expected value and variance of the contractor payment process $A_1(t)$. They are expressed as functions of time t. These quantities represent the contractors position throughout the warranty period which terminates at time T^*.

In the deterministic case the derivation of $E[A_1(t)]$ and $\mathrm{Var}[A_1(t)]$ is immediate, yielding:

$$\begin{aligned} E[A_1(t)] &= 0 , && t < s \\ &= c - a_o , && t \geq s \text{ and } s < t' \\ &= c - a_o - b(t-t'), && T^* \geq t \geq s \text{ and } s \geq t' \end{aligned}$$

$$\mathrm{Var}[A_1(t)] = 0 \quad \text{for all } t.$$

The action of the incentive fee pool is obvious in this case. So long as $a_o + b(T^* - t') < c$ failure will cost the contractor money. It will cost less when it occurs further along in the warranty period. This may be considered the classical situation from the customers point of view. However the cost reduction, without an actual profit motive might not be enough to make such a contract attractive to the contractor. On the other hand there is no risk assumption

in this simple case. The failure time is known exactly and therefore specifies a known payment requirement. Since the opportunity for profit should imply a degree of risk assumption on the part of the contractor it would be meaningless to include such a possibility ($a+b(T^*-t') > c$) in this simple case. When the failure time is uncertain, as it most often is in practice, more possibilities exist.

Though this simple case is meant only as a limit consideration and an introduction to the analysis approach, it can be thought of as a practical model in one sense. When preventive replacement is carried out at assigned times that are short enough to essentially insure no failures, the model represents the situation by interpreting deterministic failure as scheduled preventive replacement.

Section 2.2 Uniform Failure

In this section the failure time is determined by a uniform distribution having density function:

$$f_T(t) = \frac{1}{s-r}, \quad r \leq t \leq s$$

$$= 0 \quad , \quad \text{elsewhere.}$$

The analysis differs in details of calculation and results as the uniform density is placed so as to completely proceed, include, or completely follow the incentive fee parameter t'. Each of these three cases will be treated individually.

The calculations use the general formulas given at the beginning of this chapter.

Case 1 $s < t'$

Direct calculation yields:

$$E[A_1(t)] = 0 \quad , \quad t \le r$$

$$= \int_r^t (c-a_o)\frac{1}{s-r}\,dv = (c-a_o)\frac{t-r}{s-r}, \quad r \le t \le s$$

$$= \int_r^s (c-a_o)\frac{1}{s-r}\,dv = (c-a_o), \quad T^* \ge t \ge s$$

$$E[A_1^2(t)] = 0 \quad , \quad t \le r$$

$$= (c-a_o)^2\frac{t-r}{s-r}, \quad r \le t \le s$$

$$= (c-a_o)^2 \quad , \quad T^* \ge t \ge s$$

$$Var[A_1(t)] = (c-a_o)^2\frac{(t-r)(s-t)}{(s-r)^2} \quad , \quad r \le t < s$$

$$= 0 \quad , \text{ elsewhere.}$$

An alternative form of these results, particularly useful for limit studies, is obtained by the change of variables: $r = s - \epsilon$, $t = s - \beta\epsilon$, $0 \le \beta \le 1$. Only the range $r \le t \le s$ need be considered (the other values are properly assumed at the end points of this range).

$$E[A_1(t)] = (c-a_o)(1-\beta)$$

$$Var[A_1(t)] = (c-a_o)^2\,\beta(1-\beta).$$

As the width of the uniform distribution changes the expected value and variance of the payment process $A_1(t)$ are not effected. They depend only on where t is located within the range of definition of the uniform distribution. As t tends to the end value of the distribution, s, the parameter β becomes zero giving the limit values obtained for the deterministic occurance of a failure at time t=s in Section 2.1. These qualitative limit considerations will be discussed further under <u>case 3.</u>

Case 2 $\quad r < t' < s$

It is only necessary to consider the mean and variance expressions for $r \leq t \leq s$ and obtain the proper expressions for other values of t from the end point values of this range. Thus the following calculations assume the range $r \leq t \leq s$.

$$E[A_1(t)] = \int_r^t (c-a_o)\frac{1}{s-r}\,dv = (c-a_o)\frac{t-r}{s-r} \qquad , \quad r \leq t \leq t' .$$

$$E[A_1(t)] = \int_r^{t'} (c-a_o)\frac{dv}{s-r} + \int_{t'}^{t} \Big(c-a_o-b(v-t')\Big)\frac{dv}{s-r} \quad , \quad t' \leq t \leq s$$

$$= (c-a_o)\frac{t-r}{s-r} - \frac{b}{2}\frac{(t-t')^2}{(s-r)} \qquad , \quad t' \leq t \leq s$$

$$E[A_1(t)] = (c-a_o) - \frac{b}{2}\frac{(s-t')^2}{(s-r)}, \quad t \geq s .$$

The last expression is obtained by letting $t=s$ in the previous expression. In the remaining calculations it is convenient to introduce the quantity $\alpha = c-a_o + bt'$ as a combination of fee pool parameters and cost that occurs throughout much of the work. Thus the complete result is:

$$
\begin{aligned}
E[A_1(t)] &= 0 &&, \quad t \leq r \\
&= (c-a_o)\frac{t-r}{s-r} &&, \quad r \leq t \leq t' \\
&= (c-a_o)\frac{(t'-r)}{s-r} - \frac{(t'-t)}{s-r}\left[\alpha - \frac{b(t'+t)}{2}\right] &&, \quad t' \leq t \leq s \\
&= (c-a_o) - \frac{b}{2}\frac{(s-t')^2}{s-r}, &&\quad t \geq s.
\end{aligned}
$$

Calculations also yield:

$$E[A_1^2(t)] = 0 \qquad , \quad t \le r$$

$$= (c-a_o)^2 \frac{t-r}{s-r} \qquad , \quad r \le t \le t'$$

$$= (c-a_o)^2 \frac{t'-r}{s-r} - \frac{(t'-t)}{s-r}\left(\alpha^2 - \alpha b(t+t')\right) - \frac{b^2(t'^3-t^3)}{3(s-r)} \quad , \quad t' \le t \le s$$

$$= (c-a_o)^2 \frac{t'-r}{s-r} - \frac{(t'-s)}{s-r}\left(\alpha^2 - \alpha b(s+t')\right) - \frac{b^2(t'^3-s^3)}{3(s-r)} \quad , \qquad t \ge s$$

These values lead to the following expressions for the variance:

$$\mathrm{Var}[A_1(t)] = 0 \qquad , \quad t \le r$$

$$= (c-a_o)^2 \frac{(t-r)(s-t)}{(s-r)^2} \qquad , \quad r \le t \le t'$$

$$= (c-a_o)^2 \frac{t'-r}{s-r} - \frac{(t'-t)}{s-r}\left(\alpha^2 - \alpha b(t+t')\right) - \frac{b^2(t'^3-t^3)}{3(s-r)}$$

$$- \left[(c-a_o)\frac{(t-r)}{s-r} - \frac{b}{2}\frac{(t-t')^2}{(s-r)}\right]^2 \quad , \quad t' \le t \le s$$

For $t \ge s$ the expression is obtained from the last formula with t-s.

<u>Case 3</u> $r \ge t'$

In this case there is no probability of failure during the constant fee pool period $t \le t'$ so that there is an increasing incentive throughout the entire range of definition of the uniform failure time distribution. This is the most interesting of the three cases considered, and a more extensive analysis, including some limit studies will be given for it. The basic results for this case follow from application of the general formulas, yielding the following:

$$E[A_1(t)] = 0 \qquad , \quad t \le r$$

$$= \left(\alpha - \frac{b(t+r)}{2}\right)\frac{t-r}{s-r} \quad , \quad r \le t \le s$$

$$= \alpha - \frac{b(s+r)}{2} \qquad , \quad s \le t \le T^*$$

where $\alpha = c - a_o + bt'$.

$$E[A_1^2(t)] = 0 \quad , \quad t \le r$$

$$= \left[\alpha^2 - b\alpha(t+r) + \frac{b^2}{3}(t^2 + tr + r^2)\right]\frac{t-r}{s-r} \quad , \quad r \le t \le s$$

$$= \alpha^2 - b\alpha(s+r) + \frac{b^2}{3}(s^2 + sr + r^2) \quad , \quad s \le t \le T^*$$

These values yield the expression for variance:

$$Var[A_1(t)] = 0 \quad , \quad t \le t$$

$$= \frac{t-r}{(s-r)^2}\left\{\left[\alpha^2 - \alpha b(t+r)\right](s-t) + \frac{b^2}{12}\left[-3t^3 + (4s - 7r)t^2 + (4sr - r^2)(t+r)\right]\right\} , \quad r \le t \le s$$

$$= \frac{b^2}{12}(s-r)^2 \quad , \quad s \le t \le T^*$$

The end range values, $s \le t \le T^*$ can be obtained in an alternative, simple way using the properties of the uniformly distributed failure time T. This simple, direct calculation is included here for its interest and to illustrate the underlying role of the failure time process.

$$A_1(t) = 0 \quad , \quad T > t$$

$$A_1(t) = \alpha - bT \quad , \quad T \le t$$

$$E[A_1(t)] = E[\alpha] - bE[T] \quad \text{for} \quad s \le t \le T^*$$

the relation between the running time value t and the random time to failure T are such as to cause this relation to only apply in the specified time range.

Since, for the uniform random variable T, $E[T] = \frac{s+r}{2}$, and $Var[T] = \frac{(s-r)^2}{12}$, the following results are obtained

$$E[A_1(t)] = \alpha - \frac{b(s+r)}{2}$$

$$Var[A_1(t)] = Var[\alpha] + b^2\, Var[T]$$

$$= b^2\,\frac{(s-r)^2}{12}$$

as given above from the direct calculation approach.

An alternative form, useful for limit considerations is obtained by making the change of variables: $r=s-\epsilon$ and $t=s-\beta\epsilon$ where $0 \leq \beta \leq 1$. Only the mid-range expression for $r \leq t \leq s$ will be presented since the other values may be obtained from these as special cases.

$$E[A_1(t)] = \left[\alpha - bs + b(1-\beta)\frac{\epsilon}{2}\right](1-\beta)$$

$$\mathrm{Var}[A_1(t)] = (1-\beta)\left\{\left[\alpha^2 - \alpha b\left(2s - (1+\beta)\epsilon\right)\right]\beta + \frac{b^2}{12}\left[\left(3\beta^3 + 7\beta^2 + \beta+1\right)\epsilon^2 - 12s\beta(\beta+1)\epsilon + 12\beta s^2\right]\right\}$$

In this formulation ϵ represents the interval in which a failure takes place. It may be considered very small in order to yield an interesting special form that allows very detailed consideration of how the payment relates to the time at which it is considered. The simplest expression corresponding to small ϵ is to set $\epsilon=0$ in the above formulas resulting in:

$$E_L[A_1(t)] = (\alpha - bs)(1 - \beta)$$

$$\mathrm{Var}_L[A_1(t)] = (1 - \beta)\beta(\alpha - bs)^2$$

where the subscript L denotes the extreme value for $\epsilon=0$.

It is seen that $E[A_1(t)]$ and $\mathrm{Var}[A_1(t)]$ do not tend to a limit in the mathematical sense as $\epsilon \to 0$. The limit type expressions give different values for various β. From a limit point of view $E[A_1(t)]$ and $\mathrm{Var}[A_1(t)]$ depend on two variables: the width ϵ and the position of t within the distribution, determined by β. In a gross sense, $\epsilon \to 0$ forces $t \to s$ for all values of β. However the above results show that the limit depends on the values assumed for β. If $\beta=0$ the evaluation time t is held at s, corresponding to the values obtained for the deterministic situation of Section 2.1. In this case one is

considering the extreme form of the payment function after the failure has occurred, there being no density of probability left over, after time t, to allow the failure to occur later. On the other extreme with $\beta=1$ the time is considered before the probability density starts so that the failure could not have occurred and of course $E[A_1(t)] = 0$ in this case. In both of these extreme β cases the variance is zero, as it should be because there is no variation of the failure event. In the $\beta=0$ extreme the event is sure, in the $\beta=1$ extreme the event does not occur. Between these extremes the evaluation time is within the probability density (strictly speaking ϵ is small, having little effect, but not zero; however the simpler formulas, with $\epsilon=0$ are useful as approximations to the more general expressions). In such cases there is a chance that the failure has occurred by time t. There is also a chance that failure will occur after time t. The expected payment increases with t because the chance of failure, the event requiring a payment, increases with t corresponding to $\beta \rightarrow 0$. On the other hand the variance is greatest in the mid-range, falling off at each extreme because the actual occurrence of the failure event is most uncertain in the mid-range.

These are qualitative considerations expressed in mathematical terms and measurable to a degree by the quantitative parameter β. In applications they may give some guidelines on the level of resources (represented here by payments) that should be made available at a time t within a relatively short period wherein its expenditure will be required (for example: such situations can arise from scheduled replacement/repair, or high stress use of equipment at a specified time).

A particular instance of these extreme forms takes place when $s=T^*$, the end of the warranty period. This may be called the One-Hoss Shay (OHS) case after "The Deacon's Masterpiece, or, the Wonderful 'One-Hoss Shay'",

by Oliver Wendell Holmes, in which the Shay operates perfectly until, all at once it completely fails. The formulas for this situation, an ideal from the point of view of the incentivised warranty, are:

$$E_{OHS}[A_1(t)] = (\alpha - bT^*)(1-\beta)$$

$$Var_{OHS}[A_1(t)] = (1-\beta)\beta(\alpha - bT^*)^2.$$

For completeness, and as a check on some of the calculations, it is interesting to compare results for case 2 and case 3. When $t' = r$ both cases should yield the same results. This is the case with:

$$E[A_1(t)] = \left[c - a_o - \frac{b}{2}(t-r)\right]\frac{t-r}{s-r} \quad \text{where} \quad t' = r$$

in the fee pool form.

The common value of $E[A_1^2(t)]$ is $\left[\alpha^2 - \alpha b(t+r) + \frac{b^2}{3}(t^2 + tr + r^2)\right]\frac{t-r}{s-r}$ and since the expected values and expected values of the squares are equal it follows that the variance expressions are also equal as desired.

It may be noted that an interesting situation is obtained by setting $t' = 0$. This may be called the increasing incentive type of warranty having no initial period of constant (possibly zero) fee pool. Though an important special case for practical consideration it presents no special form for the expressions $E[A_1(t)]$ or $Var[A_1(t)]$ other than to observe that in this case $\alpha = c - a_o$.

Section 2.3 Exponential Failure

In the models described in this section the incentive fee pool is the same, linear segment, type used in previous sections. The failure time probability density function has the form:

$$f_T(t) = \lambda e^{-\lambda t} \quad , \quad t \geq 0$$
$$= 0 \qquad \text{elsewhere} .$$

This kind of failure time distribution gives a very different kind of model than those resulting from uniform failure time distributions. The structure is simpler, requiring less cases for consideration and may reflect some failure situations more realistically than the uniform densities could. However the exponential failure form does not allow a detailed study of the payment process such as can be obtained from placing uniform densities of selected width at various positions in the warranty period. The two kinds of distributions complement each other in the study of the incentive warranty concept. Though other types of failure distributions might be employed for practical application only these two are given detailed analysis in the report. Some discussion of general formulations is given in Chapter 4.

One difference from the uniform cases of Section 2.2 is that there is always some non-zero probability of the failure event occurring after the end of the warranty period, designated by T^*. Therefore the expected values are computed as functions of the evaluation time t, never yielding the full expectation over all non-zero failure events. The payment process $A_1(t)$ ends at T^* and by setting it to zero for $t > T^*$ one can interpret the expected value formulas (including variance) given below as being completed by the time t reaches T^*. In this interpretation, though the expected values are completed the probability of zero payment is not zero but is equal to the probability density remaining after the expiration of the warranty. Denote the probability of zero payment by P_o then:

$$P_o = \int_{T^*}^{\infty} \lambda e^{-\lambda t} dy = e^{-\lambda T^*}$$

this is a useful quantity in the exponential failure models. A similar quantity can be defined in any model assigning non-zero probability density to time events

after T^*. In studying uniform distributions this situation was not considered in the present study since it is reasonably represented in the present model.

$$E[A_1(t)] = \int_0^t (c-a_o)\,\lambda\, e^{-\lambda v}dv \qquad , \qquad 0 \le t \le t'$$

$$= \int_0^{t'} (c-a_o)\,\lambda\, e^{-\lambda v}dv + \int_{t'}^{t} (\alpha-bv)\,\lambda\, e^{-\lambda v}dv, \quad t' \le t \le T^*$$

These values become:

$$E[A_1(t)] = (c-a_o)\,(1-e^{-\lambda t}) \;\; , \quad 0 \le t \le t'$$

$$= (c-a_o)\,(1-e^{-\lambda t}) + b(t-t')\,e^{-\lambda t} + \frac{b}{\lambda}\,(e^{-\lambda t} - e^{-\lambda t'}), \; t' \le t \le T^* \; .$$

It may be observed that the expressions for $E[A_1(t)]$ agree at $t=t'$, as they should.

The term $(c-a_o)(1-e^{-\lambda t})$ is the expected contractor cost with only a constant incentive a_o provided by the customer. The remaining expressions represent the contribution of the increasing incentive pool values to the contractors payment process under its warranty obligations. If the incentive is to be effective these terms should be negative for all $t > t'$ thereby reducing payment that may occur at time before the evaluation time t. The sign of these terms is determined by the sign of the expression:

$$\lambda(t-t') + 1 - e^{-\lambda(t'-t)}$$

which may be shown to be negative for all t as required. Moreover the expression becomes increasingly negative as t increases as it should for proper incentive sturcture.

The expression for $E[A_1^2(t)]$ is as follows:

$$E[A_1^2(t)] = (c-a_o)^2 (1-e^{-\lambda t}) , \qquad 0 \le t \le t'$$

$$= (c-a_o)^2 (1-e^{-\lambda t'}) - \left[(\alpha-bt-\frac{b}{\lambda})^2 + \frac{b^2}{\lambda^2}\right] e^{-\lambda t}$$

$$+ \left[(\alpha-bt'-\frac{b}{\lambda})^2 + \frac{b^2}{\lambda^2}\right] e^{-\lambda t'} , \quad t' \le t \le T^*$$

This expression indicates a complicated form for $Var[A_1(t)]$ in the exponential case. Because of this complexity the variance is not formulated in this study. However the special case where $t'=0$, with no constant fee pool period may be considered. This is the true incentive case of most interest and the results, while not particularly simple, are reasonable.

In the case $t'=0$, $\alpha=c-a_o$ and there is only one range of definition, $(0, T^*)$, for the functions developed.

$$E[A_1(t)] = (c-a_o-\frac{b}{\lambda}) - (c-a_o-\frac{b}{\lambda}-bt)e^{-\lambda t}$$

$$E[A_1^2(t)] = (c-a_o-\frac{b}{\lambda})^2 + \frac{b^2}{\lambda^2} - \left[(c-a_o-\frac{b}{\lambda}-bt)^2 + \frac{b^2}{\lambda^2}\right] e^{-\lambda t}$$

$$Var[A_1(t)] = \frac{b^2}{\lambda^2} + \left[(c-a_o-\frac{b}{\lambda})^2 - b^2t^2 - \frac{b^2}{\lambda^2}\right] e^{-\lambda t}$$

$$- (c-a_o-\frac{b}{\lambda}-bt)^2 e^{-2\lambda t}$$

Consider the end value, at $t=T^*$, which measures the expected payment if a failure occurs within the warranty period. For simplicity the case $t'=0$ is presented:

$$E^* \equiv E[A_1(t)] = (c-a_o-\frac{b}{\lambda})(1-e^{-\lambda T^*}) + bT^* e^{-\lambda T^*}$$

If instead of a warranty period of duration T^* a lifetime maintenance was assumed by the contractor, with the same kind of incentive pool and one failure

assumed, the resulting expected payment will be:

$$E^{\infty} = c - a_o - \frac{b}{\lambda} .$$

Of course it is unrealistic to have a continually increasing, unbounded fee pool. However such an assumption is useful here for simplicity and is appropriate because the probability of failure decreases strongly as time increases. The combined effect is for a reasonable expected value, as given above.

The quantity $E^{\infty} - E^{*}$ represents the difference in payments by the contractor under lifetime maintenance and warranty type contracts. Of course the major assumptions are at most one failure, and the same incentive fee pool. Though somewhat unrealistic as practical contract assumptions, this illustration gives an idea of one methodological approach for comparing different maintenance concepts.

$$E^{\infty} - E^{*} = [b T^{*} - (c - a_o - \frac{b}{\lambda})] e^{-\lambda T^{*}},$$

and depending on the parameter values different situations may be considered. A case of some interest occurs when the factors effecting the payment: failure distribution, and fee pool form, are such as to give the same expected payment with both methods. This occurs when $E^{\infty} - E^{*} = 0$ or:

$$b T^{*} = c - a_o - \frac{b}{\lambda}$$

A result such as this indicates the interrelation of contract parameters, a_o, b, T^{*}, replacement cost c, and failure distribution parameter λ. To the extent that such expressions could be developed (even in tabular form for complicated situations) to represent actual situations, they would be useful negotiation tools.

Section 2.4 Discussion, Including Present Value of Money

The payment by the contractor under an incentive warranty is a stochastic process of simple type, defined as:

$$
\begin{aligned}
A_1(t) &= 0 && , \quad T > t \\
&= c - a_o && , \quad T \leq t', \quad T \leq t \leq T^* \\
&= c - a_o + bt' - bT && , \quad T > t', \quad T \leq t \leq T^* \\
&= 0 && , \quad T > T^* .
\end{aligned}
$$

The actual payment is, in any case, a numerical value determined by the time at which the random failure event takes place, denoted by the random variable T. Thus $A_1(t)$ is a step function of t with the step location determined by T and the step height determined by the fee pool value at the time the step occurs.

In motivating the contractor to increase the time to failure by providing an increasing fee pool there is a danger that the contractor will desire a failure toward the end of the warranty period so as to receive incentive fee. This undesirable situation can be prevented by selecting contract parameters so that $c-a_o + bt' - bT^* > 0$ so that any failure will actually cost the contractor money.

There is another point of view however that considers the provision of an incentive warranty fee pool insufficient inducement for the contractor to agree to a warranty. It is true that its risk of payment is reduced, but with the condition of possible negative payment the warranty still represents a (randomly specified) cost to the contractor. Therefore there is a problem with the basic approach: in order to avoid the counter motivation of wanting a failure the chance for obtaining profit by agreeing to a warranty is lost. One way to combine the desired motivation is to agree to an award fee of some kind if no failure occurs by the end of the warranty period. A reasonable value for such an award fee might be $a_o - bt' + bT^*$, the value reached by the incentive

fee pool by the end of the warranty period. If availability is a major concern to the customer, with savings in money a secondary consideration, then a substantial award fee suggests itself. In the combined incentive warranty pool, award fee concept the contractor is motivated to assume the warranty obligation by the chance of profit and motivated to design for long operating times (in the at most one failure case).

In the previous sections of this chapter the payment process $A_1(t)$ is characterized by the expected payment up to time t and by the variance of the process. These expressions may be useful for planning before the start of the warranty period and as management decision tools during the warranty period. Of course once the failure occurs the payment is known exactly and the process $A_1(t)$ ceases to represent the payment. The effectiveness of the models and their formulas for expected value and variance for planning and control purposes depends to some extent on the time scales involved. When the warranty period is long the change in expected payment over time can be a valuable planning tool, particularly when the necessary payments under the warranty are large.

The research described in this report is directed toward developing a methodology for studying the concepts of incentivising warranties. However some indication of the application of this methodology can be found in the following discussion. As an example case the third case of Section 2.2 is considered. In this case the uniform distribution is used with $r \geq t'$, so actual payment must occur in the purely increasing incentive region of the fee pool

The variance is a cubic function of t with complicated coefficients. It is best studied by numerical calculations (graphic plots) which are not included as part of the present theoretical research. However the expected value function, denoted by $E(t) = E[A_1(t)]$ below, is a relatively simple quadratic function of t of the form:

$$E(t) - A = - B(t - C)^2$$

where $\quad A = \frac{b}{2(s-r)} \left(r - \frac{\alpha}{b}\right)^2, \qquad B = \frac{b}{2(s-r)}, \qquad C = \frac{\alpha}{b}.$

This function is a parabola in the $(t, E(t))$ plane with vertex at (C,A), crossing the time axis at r and $\frac{2\alpha}{b} - r$. As discussed above, a reasonable condition for the incentive warranty parameters is that $\alpha - bt \geq 0$ for all $t \leq T^*$ which implies that $\frac{\alpha}{b} \geq T^*$ in any warranty situation satisfying this condition. Under this assumption the function $E(t)$ is increasing over the portion of the warranty period in which the failure takes place, the interval (r, s) with $s < T^*$. The maximum value is reached when $t=s$ and from then on $E^*(t) = \alpha - \frac{b(s+r)}{2}$ for $s \leq t \leq T^*$. This value is easily shown to be less than the maximum of the parabolic function $E(t)$ continued beyond $t=s$. Therefore the relations between the various time events, and expected values are as shown in Figure 2.1, where $E(t)$ is the quadratic function and $E^*(t)$ the actual expected value function, $E^*(t) = E(t)$, $r \leq t \leq s$.

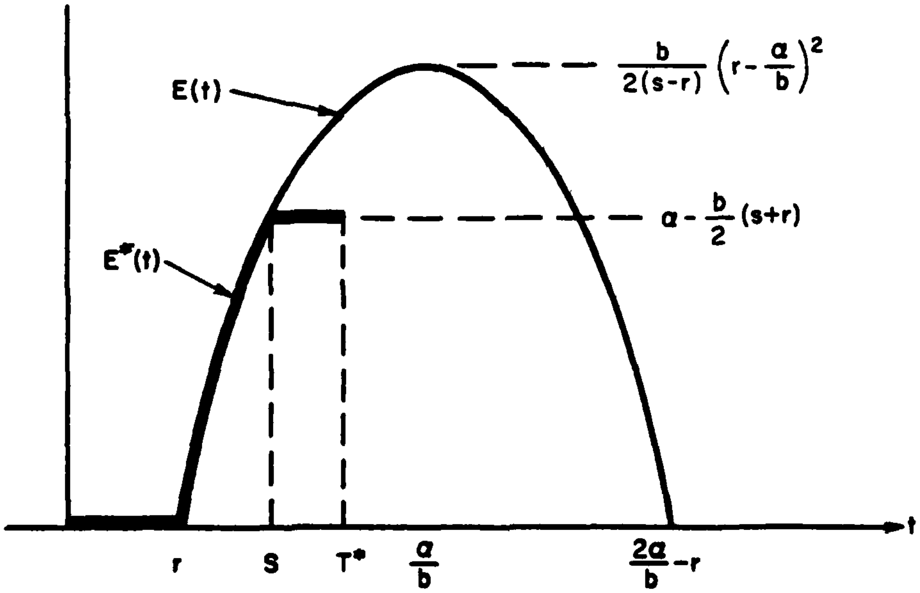

Figure 2.1 Expected value function for Case 3 Section 2.2.

The actual payment, as distinct from its expected value function, is less when the time of the failure event is great. The least payment is at $t = T^*$ in amount $\alpha - bT^*$ and the greatest payment is at $t=0$ in amount α. However in the case under consideration here the failure must occur in the interval (r, s) with $t' < r < s < T^*$ so the maximum payment in this model is $\alpha - br$ and the minimum payment is $\alpha - bs$. The average of these values equals the value of the expected value function for $s \leq t \leq T^*$. Thus the contractor may pay as little as $\alpha - bs$ or as much as $\alpha - br$ in a particular situation. If the same kind of contract was repeated many times the average payment over all would be $\alpha - \frac{b}{2}(s+r)$ per contract. This average value is the value actually paid when failure occurs at the average time $(r+s)/2$. Actual payment is contrasted with the expected value function in Figure 2.2 for the range (r, s) in which the actual payment must be determined.

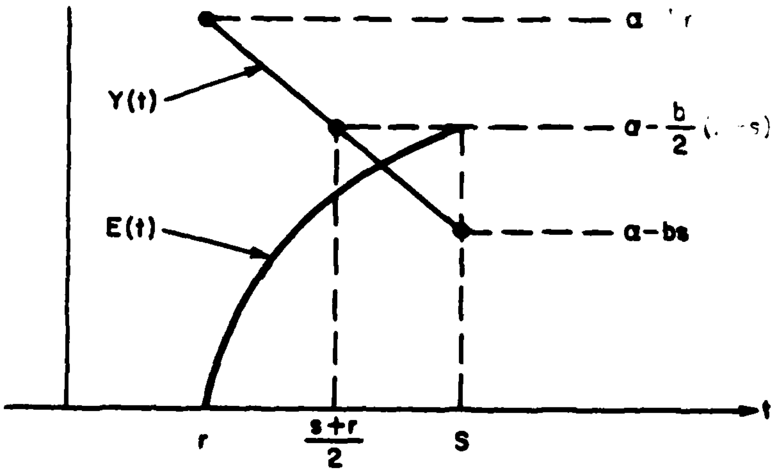

Figure 2.2 Actual and Expected Payment.

In Figure 2.2, $Y(t)$ represents the actual payment if the failure event occurs at time t, i.e. $T=t$. Though these are events of probability zero due to the continuous distribution assumed for the random variable T, the curve $Y(t)$ still serves as an indication of what the costs will be should failure occur at time t.

An alternative consideration can be made utilizing cumulative probability statements about the amount of payment. For example if the random variable Y is used to denote payment then $P[T \leq \tau] = P[Y \leq \alpha - b\tau]$. The probability distribution yields:

$$P[Y \leq \alpha - b\tau] = \frac{\tau - r}{s - r} .$$

Therefore if it is desirable to keep the warranty payment below a value $\alpha - b\tau$ the equipment must be designed to have a time to failure exceeding τ. The strength of this desire may be measured by the probability $\frac{\tau - r}{s - r}$. In this situation the contractor can spend money on improving the reliability so that failure occurs after time τ or he can take the risk, measured by $1 - \frac{\tau - r}{s - r} = \frac{s - \tau}{s - r}$, that the failure will occur after τ with the (r, s) uniform failure design. These considerations illustrate the interplay between the incentive warranty methodology and equipment reliability in which various management decisions and design tradeoffs exist.

In all of the discussion above the present value of future expenditures of money was not considered. The purpose of the research in this report is to introduce the concepts and model methodology for incentive warranties. In such considerations it is not necessary to include the actual value of money (or resources) such details falling into the application area based upon the theoretical developments of this study. However to illustrate how the value of money can be introduced into the methodology an example will now be considered.

Using discount rate d a discount factor of $(1-d)^{\tau}$ is applied to an expenditure of money (resources) at time t to yield the present values of that expenditure, as discussed in Section 1.3. This consideration can be applied to the payment itself or to its expected function. Alternatively various "expected value" approaches can be taken to give (simpler) approximate results. Such

expected value techniques are widely used in the analysis of random models. Therefore it is interesting to consider several formulations of the present value analysis. The uniform distribution, Case 3 with $t' < r$ will continue to serve as the example model.

The present value, considered as a random available, has the form $V = (\alpha - bT)(1-d)^T$, where T is the uniformly distributed random variable in the interval (r,s). This formulation shows the disadvantage of a discount factor of the given type for calculations such as expected value. Therefore an alternative discount factor e^{-ht} is used, where $e^{-h} = 1 - d$ relates the two discount rate forms.

Now the present value is the random variable:

$$V = (\alpha - bT)e^{-hT}.$$

Direct calculation yields:

$$E[V] = \frac{1}{h(s-r)}\left[\left(bs + \frac{b}{h} - \alpha\right)e^{-hs} + \left(\alpha - br - \frac{b}{h}\right)e^{-hr}\right]$$

This is to be compared with the expected payment without discounting $E[Y] = \alpha - \frac{b}{2}(s+r)$. By expanding the exponential factors in $E[V]$ one can write:

$$E[V] = \frac{1}{(s-r)}\left[\left(\frac{bs-\alpha}{h} + \frac{b}{h^2}\right)\left(1 - hs + \frac{h^2s^2}{2} + 0(h^3)\right)\right.$$

$$\left. + \left(\frac{\alpha - br}{h} - \frac{b}{h^2}\right)\left(1 - hr + \frac{h^2r^2}{2} + 0(h^3)\right)\right]$$

In this form it is possible to obtain the limit as the discount rate goes to zero. The result is as expected:

$$\lim_{h\to 0} E[V] = E[Y] \quad .$$

At the other extreme for total discounting the discount rate d=1 with

corresponding condition obtained by taking the limit as h becomes unbounded. This clearly yields $\lim_{h \to \infty} E[V] = 0$ as expected.

For completness two expected value type models will be given next. They illustrate techniques that are sometimes used and may give results ranging from correct to approximate to totally wrong and misleading.

Expected value model A:

$$E_A = E[\alpha - bT]\, e^{-hE[T]}$$

$$E_A = \left[\alpha - \frac{b}{2}(s+r)\right] e^{-\frac{h}{2}(s+r)}$$

Expected value model B:

$$E_B = E[\alpha - bT]\, E\left[e^{-hT}\right]$$

$$= \frac{1}{h(s-r)} \left(\frac{b}{2}(s+r) - \alpha\right) e^{-hs} + \left(\alpha - \frac{b}{2}(s+r)\right) e^{-hr}$$

It can be seen that neither of these procedures give the value $E[V]$ and the question of their value as approximations would require numerical comparison of the forms as functions of h. However it is interesting to note that both expected value models give the non-discounted expected value in the limit as $h \to 0$ and both tend to zero as h becomes unbounded. Thus their gross behavior follows the correct value $E[V]$ in the extreme cases. Further investigation of the value of these quantities as useful approximations to $E[V]$ is outside the scope of the work covered by this report. It should be observed that one would expect that $E[V] \leq E[Y]$ for all values of h, equality holding only when h=0. For the expected value models to be at all useful they too should satisfy this condition with respect to the undiscounted payment expression. Analytic varification of these conditions seems excessively complicated

and numerical checks would be the most direct way of varifying (or disproving) the relation. Such numerical work is outside the scope of this initial, largely theoretical, conceptual study.

Chapter 3 At Most Two Failures

Section 3.1 Formulation of the Model

When more than one failure can occur many new features may be introduced into the warranty payment model. The material in this report represents an initial formulation of the incentive warranty structure and as such it treats rather simple models. The cost of repair is considered a constant c for each failure, no consideration is given to time to repair/replace, nor are logistic support features such as preventative maintenance included. Moreover the forms of the incentive pool and failure time probability are simple. This chapter considers the situation in which at most two failures can occur. The model is considered from various points of view and the payment process $A_2(t)$ is taken as the mathematical model for the incentive warranty structure. The expected value and variance of $A_2(t)$ are calculated as being the representational quantities of interest. However it should be noted that one could also consider the probability that $A_2(t)$ has some particular property e.g. $P[A_2(t) \leq k]$ might prove to be of value in the study of process $A_2(t)$. The present report does not work with such probability values, restricting itself to development of the representational moment expressions.

In this section the basic assumptions will be given, the mathematical model formulated, and general expressions for expected value and variance stated.

To model the incentive warranty process the most basic quantity required is the form of the incentive fee pool. In the case of more than one failure there is a complication with the idea of incentive payment. If the maximum payment is less than c, the cost of a single repair then there is no difficulty. However such a value for incentive may not be considered sufficient

motivation by the contractor for assuming the warranty obligation. Therefore, in the at most two failure case it seems reasonable to have the maximum exceed c but be less than 2c. In such a situation, if the first failure occurs after a long enough period the fee pool may exceed c at the time of failure. Then one must postulate what is to be done: the contractor may take up to c and return the extra amount to the pool, or lose the extra amount, or take the extra amount as a sort of milestone profit. It is likely that a study of these alternatives would produce information useful in the structuring of actual incentive warranties, they can be made even more realistic (and involved) by including considerations about the value of money. However for the present research the simplest kind of model was used in which the contractor receives the fee pool amount specified by the incentive fee function at the time of failure. At that time the fee pool starts over at zero and increases at the same rate. For some contract parameter values the payment could go negative in such a model, representing a profit for the contractor. This situation could motivate the contractor to assume the warranty and to produce equipment with a desirable availability profile. However such an instrument may also improperly motivate the contractor, particularly if parameter values are negotiated so as to give an advantage to the contractor. Thus the present research deals with a rather simple mathematical model form that contains, by implication some important contractual considerations.

This incentive fee function is assumed to be a linear function having constant slope b. If T_1 and T_2 represent the random variables: time to first and second failures, respectively, then the payment process for at most two failures has the form:

$$A_2(t) = c - bT_1 + c - b(T_2 - T_1).$$

Alternatively if $R_1 = T_1$ and $R_2 = T_2 - T_1$ are the random failure times to the first failure and from the first to the second failure, respectively, the process is expressed as:

$$A_2(t) = c - b\,R_1 + c - b\,R_2.$$

The random variables R_1 and R_2 are commonly used in availability work and will form the two random variable approach used in much of this chapter.

Let $f_{R_1R_2}(r_1, r_2)$ be the joint probability density function for the random variables R_1 and R_2 then:

$$E[A_2(t)] = \int_0^t \int_0^{t-r_2} \left[2c - b(r_1 + r_2)\right] f_{R_1R_2}(r_1, r_2)\, dr_1\, dr_2$$

$$E[A_2(t)] = 2c\, P\left[R_1 + R_2 \leq t\right] - b \int_0^t \int_0^{t-r_2} (r_1 + r_2) f_{R_1R_2}(r_1, r_2)\, dr_1\, dr_2.$$

A similar formulation may be written for $E[A_2^2(t)]$ from which the $\mathrm{Var}[A_2(t)]$ can be obtained using $\mathrm{Var}[A_2(t)] = E[A_2^2(t)] - \left(E[A_2(t)]\right)^2$.

In the rest of this chapter R_1 and R_2 are assumed to be independent, exponentially distributed random variables so that:

$$f_{R_1R_2}(r_1, r_2) = \lambda_1 \lambda_2\, e^{-\lambda_1 r_1}\, e^{-\lambda_2 r_2}\ .$$

3.2 Equal Failure Rates

In this section the expected value and variance of the payment process are obtained for the special case in which $\lambda_1 = \lambda_2 = \lambda$. The two variable R_1, R_2, representation of $A_2(t)$ is used. For convenience in calculation let:

$$H(t) = \int_0^t \int_0^{t-r_2} (r_1 + r_2)\, f_1(r_1)\, f_2(r_2)\, dr_1\, dr_2$$

where the marginal distributions are $f_1(r_1) = \lambda_1 e^{-\lambda_1 r_1}$, and $f_2(r_2) = \lambda_2 e^{-\lambda_2 r_2}$.

Direct calculation yields:

$$P\left[R_1 + R_2 \leq t\right] = 1 - (1 + \lambda t)\, e^{-\lambda t}$$

$$H(t) = 2\left[\frac{1}{\lambda} - \left(\frac{1}{\lambda} + t + \frac{\lambda t^2}{2}\right) e^{-\lambda t}\right]$$

Thus:

$$E[A_2(t)] = 2c\, P\left[R_1 + R_2 \leq t\right] - b\, H(t)$$

$$E[A_2(t)] = 2c - \frac{2b}{\lambda} + \left[2b\left(\frac{1}{\lambda} + t + \frac{\lambda t^2}{2}\right) - 2c\,(1 + \lambda t)\right] e^{-\lambda t} \quad .$$

If the expected value is taken for the full range of t (the positive real axis) the value is $2c - \frac{2b}{\lambda}$ which agrees with the above expression in the limit as $t \to \infty$.

The calculation of $E[A_2^2(t)]$ is more involved and proceeds as follows:

$$E[A_2^2(t)] = 4c^2\, P\left[R_1 + R_2 \leq t\right] - 4bc\, H(t) + b^2\, k(t)$$

where $P\left[R_1 + R_2 \leq t\right]$ and $H(t)$ are given above. The quantity

$$k(t) = \int_0^t \int_0^{t-r_2} (r_1^2 + 2r_1 r_2 + r_2^2)\, \lambda^2\, e^{-\lambda(r_1 + r_2)}\, dr_1\, dr_2$$

Evaluation of this integral yields:

$$k(t) = 3\left[\frac{2}{\lambda^2} - \left(\frac{2}{\lambda^2} + \frac{2t}{\lambda} + t^2 + \frac{\lambda t^3}{3}\right) e^{-\lambda t}\right]$$

Therefore:

$$E[A_2^2(t)] = 4c^2\left[1 - (1+\lambda t)e^{-\lambda t}\right] - 8bc\left[\frac{1}{\lambda} - \left(\frac{1}{\lambda} + t + \frac{\lambda t^2}{2}\right)e^{-\lambda t}\right]$$
$$+ 3b^2\left[\frac{2}{\lambda^2} - \left(\frac{2}{\lambda^2} + \frac{2t}{\lambda} + t^2 + \frac{\lambda t^3}{3}\right)e^{-\lambda t}\right]$$

Using this value together with $\left(E[A_2(t)]\right)^2$ gives:

$$\mathrm{Var}[A_2(t)] = \frac{2b^2}{\lambda^2} + \left[4c^2 - \frac{8bc}{\lambda} + \frac{2b^2}{\lambda^2} + \left(4c^2\lambda - 8bc + \frac{2b^2}{\lambda}\right)t\right.$$
$$\left. + b^2t^2 - b^2\lambda t^3\right]e^{-\lambda t} + \left[- 2c(1 + \lambda t)\right.$$
$$\left. + 2b\left(\frac{1}{\lambda} + t + \frac{\lambda t^2}{2}\right)\right]^2 e^{-2\lambda t}.$$

As can be seen this rather complicated expression has a constant term equal to the overall variance (obtained here as the limit for $t \to \infty$) and two different exponential terms. The later terms have polynomials in t as coefficients. Detailed study of the expressions for $E[A_2(t)]$ and $\mathrm{Var}[A_2(t)]$ require numerical approaches outside the scope of the present study. Their complexity indicates that a corresponding complexity is to be expected in the general case where $\lambda_1 \neq \lambda_2$. The present special case serves as a check on the more general expressions developed in the next section.

3.3 Unequal Failure Rates

As indicated by the equal failure rates case considered above the expressions for $E[A_2(t)]$ and particularly for $\mathrm{Var}[A_2(t)]$ become complicated in the more general case. For this reason two different approaches are used, one, the single variable approach being simpler to carry out than the other which is the standard two variable treatment of the kind used in the calculation of the previous section.

3.3.1 The Two Random Variable Approach

Due to the complexity of calculations only the quantity $E[A_2(t)]$ is obtained by the two variable approach. Its value serves as a check on the calculations using the one variable approach and also for consistency with the one failure result $E[A_1(t)]$.

The density functions for R_1 and R_2 are $\lambda_1 e^{-\lambda_1 r_1}$ and $\lambda_2 e^{-\lambda_2 r_2}$ respectively, and

$$E[A_2(t)] = 2c\, P\left[R_1 + R_2 \leq t\right] - b\, H_2(t)$$

where
$$H_2(t) = \int_0^t \int_0^{t-r_2} (r_1 + r_2)\, \lambda_1 \lambda_2\, e^{-\lambda_1 r_1} e^{-\lambda_2 r_2}\, dr_1\, dr_2,$$

$$P\left[R_1 + R_2 \leq t\right] = \int_0^t \int_0^{t-r_2} \lambda_1 \lambda_2\, e^{-\lambda_1 r_1} e^{-\lambda_2 r_2}\, dr_1\, dr_2.$$

Direct calculation yields:

$$P\left[R_1 + R_2 \leq t\right] = 1 - e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right),$$

$$H_2(t) = \frac{1}{\lambda_2} - \left(t + \frac{1}{\lambda_2}\right) e^{-\lambda_2 t} + \frac{1}{\lambda_1} - \frac{e^{-\lambda_2 t}}{\lambda_1} + \frac{\lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1}\left(t + \frac{1}{\lambda_1}\right)\left[e^{-(\lambda_2 - \lambda_1)t} - 1\right]$$

These values give:

$$E[A_2(t)] = 2c - b\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) + \Bigg\{\left[b\left(t + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - 2c\right] e^{-\lambda_2 t}$$
$$+ \frac{2c\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) - \frac{b\lambda_2}{\lambda_2 - \lambda_1}\left(t + \frac{1}{\lambda_1}\right)\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)\Bigg\}$$

This expression can be written in several ways but all of them seem to be of about the same complexity. The direct calculations required, though simple, are involved and it does not seem desirable to obtain $E[A_1^2(t)]$ and hence the variance by the two variable method.

Due to the complexity of the results it is desirable to check them against each other. This is done in Section 3.4. However it may be noted here that the overall expected values, obtained if the process goes beyong T^* should agree with the formulas as $t \to \infty$. Let $A_2 = 2c - b(R_1 + R_2)$ denote the random variable when time is not limited by T^*. Then

$$E[A_2] = 2c - b\left(E[R_1] + E[R_2]\right)$$

$$E[A_2] = 2c - b\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right),$$

and

$$\mathrm{Var}[A_2] = b^2\left(\mathrm{Var}[R_1] + \mathrm{Var}[R_2]\right),$$

$$\mathrm{Var}[A_2] = b^2\left[\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right].$$

These values agree with the limit values for the $\lambda_1 = \lambda_2 = \lambda$ case and may be used as approximations to the actual expressions for $E[A_2(t)]$ and $\mathrm{Var}[A_2(t)]$. The value of such approximations depends in part on the probability that the failures occur after T^*. Because of the possibility of one or two failures occurring the problem of estimating the degree of approximation does not appear to be simple.

3.3.2 The One Random Variable Approach

An alternative, and in some ways simpler, approach to the study of the process $A_2(t)$ is in terms of the single random variable T_2, the time (starting from zero) to the second failure event. The reason for including

both approaches in at least part of this report is to provide optional model forms and to allow calculations based on one approach to be checked against the other for consistency.

The process $A_2(t) = c - bT_1 + c - b(T_2 - T_1)$ can be written more simply as:

$$A_2(t) = 2c - bT_2.$$

To study $A_2(t)$ in terms of the random variable T_2 the probability density function $f_{T_2}(v)$ is required. Since $T_2 = R_1 + R_2$ the previously computed value for $P[R_1 + R_2 \leq v]$, with $\lambda_1 \neq \lambda_2$, can be used to obtain the distribution function:

$$F_{T_2}(v) = P\left[R_1 + R_2 \leq v\right] = 1 - e^{-\lambda_2 v} + \frac{\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_2 v} - e^{-\lambda_1 v}\right).$$

Differentiation and arrangement of terms results in:

$$f_{T_2}(v) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 v} - e^{-\lambda_2 v}\right).$$

Partial expectations of T_2 restricting $v \leq t$ are used to obtain expectations for $A_2(t)$. The concept of partial expectations is discussed more fully in Chapter 4.

Let $E_t[T_2]$ denote the partial expectation of T_2, restricting v to: $v \leq t$. Then:

$$E_t[T_2] = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \int_0^t v\left(e^{-\lambda_1 v} - e^{-\lambda_2 v}\right) dv$$

$$E_t[T_2] = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} + t\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) + \frac{e^{-\lambda_2 t}}{\lambda_2^2} - \frac{e^{-\lambda_1 t}}{\lambda_1^2}\right]$$

Then $E[A_2(t)] = 2c\,P[T_2 \le t] - b\,E_t[T_2]$, which gives:

$$E[A_2(t)] = 2c\left\{1-e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)\right\}$$

$$- \frac{b\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left\{\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} + t\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) + \frac{e^{-\lambda_2 t}}{\lambda_2^2} - \frac{e^{-\lambda_1 t}}{\lambda_1^2}\right\}.$$

This expression can be shown to equal the expression obtained in Section 3.3.1 and in the limit $\lambda_2 \to \lambda_1 = \lambda$ to yield the expression from Section 3.2, these considerations are discussed in Section 3.4.

To obtain $E[A_2^2(t)]$ and hence $Var[A_2(t)]$ the expression $E_t[T_2^2]$ is required.

$$E_t[T_2^2] = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_0^t v^2\left(e^{-\lambda_1 v} - e^{-\lambda_2 v}\right)dv$$

$$E_t[T_2^2] = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left\{-\frac{t^2}{\lambda_1}e^{-\lambda_1 t} + \frac{2}{\lambda_1^3} - 2\left(t+\frac{1}{\lambda_1}\right)\frac{e^{-\lambda_1 t}}{\lambda_1^2} + \frac{t^2}{\lambda_2}e^{-\lambda_2 t}\right.$$

$$\left. - \frac{2}{\lambda_2^3} + 2\left(t+\frac{1}{\lambda_2}\right)\frac{e^{-\lambda_2 t}}{\lambda_2^2}\right\}.$$

$$E[A_2^2(t)] = E_t[4c^2 - 4bc\,T_2 + b^2\,T_2^2]$$

$$= 4c^2\,P[T_2 \le t] - 4bc\,E_t[T_2] + b^2\,E_t[T_2^2]$$

All the quantities required for this expression have been obtained above. By putting them into the expression it becomes:

$$E[A_2^2(t)] = D_1 + D_2 + D_3 \ , \quad \text{where}$$

$$D_1 = 4c^2\left[1 - e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)\right],$$

$$D_2 = \frac{-4bc\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} + t\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) + \frac{e^{-\lambda_2 t}}{\lambda_2^2} - \frac{e^{-\lambda_1 t}}{\lambda_1^2}\right],$$

$$D_3 = \frac{b^2\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[-\frac{t^2}{\lambda_1}e^{-\lambda_1 t} + \frac{2}{\lambda_1^3} - 2\left(t+\frac{1}{\lambda_1}\right)\frac{e^{-\lambda_1 t}}{\lambda_1^2} + \frac{t^2}{\lambda_2}e^{-\lambda_2 t}\right.$$

$$\left. - \frac{2}{\lambda_2^3} + 2\left(t + \frac{1}{\lambda_2}\right)\frac{e^{-\lambda_2 t}}{\lambda_2^2}\right].$$

These expressions can be used to give the variance in the form:

$$\mathrm{Var}[A_2(t)] = E[A_2^2(t)] - \left(E[A_2(t)]\right)^2 .$$

The only way to consider such complex expressions is as numerical functions of t. This kind of detailed calculations is outside the scope of this report. However internal checks and limit cases can reasonably be considered as is done in the next section. However the case where $t\to\infty$ can be caluclated directly. This yields:

$$E[A_2] = 2c - b\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)$$

$$E[A_2^2] = 4c^2 - 4bc\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right) + \frac{b^2\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{2}{\lambda_1^3} - \frac{2}{\lambda_2^3}\right)$$

$$= 4c^2 - 4bc\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + 2b^2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right)$$

These values give:

$$\mathrm{Var}[A_2] = b^2\left[\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right]$$ which agrees with the overall expected value and variance values obtained at the end of Section 3.3.1.

3.4 Verifications and Limit Values

3.4.1 Agreement of Approaches

In Section 3.3.1 the expression for $E[A_2(t)]$, denoted as $E_1[A_2(t)]$ was:

$$E_1[A_2(t)] = 2c - b\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) + \left\{\left[b\left(t + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - 2c\right]e^{-\lambda_2 t}\right.$$

$$\left. + \frac{2c\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) - \frac{b\lambda_2}{\lambda_2-\lambda_1}\left(t + \frac{1}{\lambda_1}\right)\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)\right\}.$$

The expression obtained in Section 3.3.2, denoted as $E_2[A_2(t)]$ was:

$$E_2[A_2(t)] = 2c\left\{1 - e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)\right\}$$

$$- \frac{b\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left\{\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} + t\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) + \frac{e^{-\lambda_2 t}}{\lambda_2^2} - \frac{e^{-\lambda_1 t}}{\lambda_1^2}\right\}.$$

It is not immediately evident that these two expressions are the same. Showing that they are the same gives an indication that the expression for $E[A_2(t)]$ is correct and that the two approaches to model formulation and analysis are equivalent. This constitutes an internal consistency check on the research.

To show that the two expressions are the same, similar terms are compared. The constant terms are both equal to:

$2c - b\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right)$. The remaining term with c as a factor is also seen to be the same in both cases, equal to:

$$2c\left[\frac{\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) - e^{-\lambda_2 t}\right].$$

The remaining term in $E_1[A_2(t)]$ is:

$$b\left(t+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)e^{-\lambda_2 t} - \frac{b\lambda_2}{\lambda_2-\lambda_1}\left(t+\frac{1}{\lambda_1}\right)\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

In $E_2[A_2(t)]$ this term corresponds to:

$$\frac{-b\,\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[t\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) + \frac{e^{-\lambda_2 t}}{\lambda_2^2} - \frac{e^{-\lambda_1 t}}{\lambda_1^2}\right].$$

A little algebra shows that both of these expressions are equal to:

$$\frac{-b}{\lambda_2-\lambda_1}\left[\lambda_1\left(t+\frac{1}{\lambda_2}\right)e^{-\lambda_2 t} - \lambda_2\left(t+\frac{1}{\lambda_1}\right)e^{-\lambda_1 t}\right].$$

Therefore it is established that $E_1[A_2(t)] = E_2[A_2(t)]$ as required for consistency. The two approaches give the same value for $E[A_2(t)]$. Since the variance was only obtained for the one variable approach no check of this kind can be made for its expression.

3.4.2 Limits as Failure Rates Become Equal

Both $E[A_2(t)]$ and $Var[A_2(t)]$ are available in the case where $\lambda_1 = \lambda_2 = \lambda$, from Section 3.2. The expressions for these quantities for the general case $\lambda_1 \neq \lambda_2$ are obtained in Section 3.3.2. These should tend to the special case values as $\lambda_1 - \lambda_2$ tends to zero. The expressions in question are simple functions of the failure rates λ_1 and λ_2 and the limit as $\lambda_1 - \lambda_2 \to 0$ certainly exists. The intent of this section is to evaluate that limit and show that it gives the desired special case value. To do this it is satisfactory to take any particular approach of the variables to their limit values. Two forms are used in the following: $\lambda_2 - \lambda_1 = v \to 0$ and $\lambda_2 = \lambda$ while $\lambda_1 \to \lambda$.

First consider the $E[A_2(t)]$ expression. Write this as: $E[A_2(t)] = 2cP - bH_2(t)$. The limits of each part will be obtained and shown to equal the corresponding P and H(t) expressions in the $\lambda_1 = \lambda_2$ case.

$$P = 1 - e^{-\lambda t} + \lambda\, e^{-\lambda t} \lim_{v \to 0} \frac{e^{-vt} - 1}{v}$$

$$P = 1 - (1 - \lambda t)\, e^{-\lambda t}$$, which is the special case expression.

Note that $\lim_{\lambda_2 - \lambda_1 \to 0} \left[\frac{-1 + e^{-(\lambda_2 - \lambda_1)t}}{\lambda_2 - \lambda_1} \right] = -t$ so that

$$\lim_{\lambda_2 - \lambda_1 \to 0} H_2(t) = \frac{1}{\lambda} - \left(t + \frac{1}{\lambda}\right) e^{-\lambda t} + \lambda\, e^{-\lambda t}\left(t + \frac{1}{\lambda}\right)(-t) + \frac{1}{\lambda} - \frac{e^{-\lambda t}}{\lambda}$$

$$= \frac{2}{\lambda} - 2\left(\frac{1}{\lambda} + t + \frac{\lambda t^2}{2}\right) e^{-\lambda t},$$ the special case value. Therefore

$\lim_{\lambda_1 - \lambda_2 \to 0} E[A_2(t)] =$ the special case value of $E[A_2(t)]$ when $\lambda_1 = \lambda_2 = \lambda$.

Similar analysis is used to show that the general case variance goes to the special form. The procedure is to show that the expression for $E[A_2^2(t)]$ has the correct limit form since it has already been established that the expected value has the proper limit. The rather involved calculations are carried out for each of the terms D_1, D_2, and D_3 used to express $E[A_2^2(t)]$ in Section 3.3.2. These may be compared individually with the corresponding terms in the special case expression for $E[A_2^2(t)]$ given in Section 3.2. Direct calculation of limits, setting $\lambda_2 = \lambda$ and letting $\lambda_1 \to \lambda$ yields:

$\lim_{\lambda_1 \to \lambda} D_1 = 4c^2\left[1 - e^{-\lambda t} - \lambda t e^{-\lambda t}\right]$, which is the correct equal failure

rate term containing c^2.

$$\lim_{\lambda_1 \to \lambda} D_2 = -4bc\left[\frac{2}{\lambda} - \left(\frac{2}{\lambda} + 2t + \lambda t^2\right) e^{-\lambda t}\right],$$ which is the correct term

containing bc.

$$\lim_{\lambda_1 \to \lambda} D_3 = b^2\left[\frac{6}{\lambda^2} - \left(3t^2 + \lambda t^3 + \frac{6t}{\lambda} + \frac{6}{\lambda^2}\right) e^{-\lambda t}\right],$$

the correct b^2 term. Thus the $\mathrm{Var}[A_2(t)]$ for the general failure rate case does become the value obtained for the equal case $\lambda_1 = \lambda_2 = \lambda$. This provides a further consistency check on the models and their analysis.

3.4.3 Limits as the Second Failure Rate Increases

If λ_2 becomes large the time between the first and second failures reduces in a probabilistic sense. In the limit as $\lambda_2 \to \infty$ the two failures occur together. In this case the results should agree with the one failure model except that the cost parameter becomes 2c corresponding to the fact that there are actually two failures, or double the cost, at the combined failure time. Evaluation of this limit case serves as a check on the single and double failure models.

Section 2.3 gives the following values for the one failure case. Parameters are adjusted so as to correspond to the model of the present Chapter ($a_o = 0$, $t' = 0$, and $c = 2c$).

$$E[A_1(t) = 2c\left(1 - e^{-\lambda t}\right) - b\left[\frac{1}{\lambda} - \left(\frac{1}{\lambda} + t\right) e^{-\lambda t}\right]$$

$$\mathrm{Var}[A_1(t)] = \frac{b^2}{\lambda^2} + \left[\left(2c - \frac{b}{\lambda}\right)^2 - b^2 t^2 - \frac{b^2}{\lambda^2}\right] e^{-\lambda t}$$

$$- \left(2c - \frac{b}{\lambda} - bt\right)^2 e^{-2\lambda t}$$

Using the expressions for $E[A_2(t)]$ and $E[A_2^2(t)]$ developed in Section 3.3.2, taking the limit as $\lambda_2 \to \infty$ yields the following

$$\lim_{\lambda_2 \to \infty} E[A_2(t)] = 2c\left[1 - e^{-\lambda t}\right] - b\lambda\left[\frac{1}{\lambda^2} - \left(\frac{1}{\lambda^2} + \frac{t}{\lambda}\right) e^{-\lambda t}\right]$$

where $\lambda_1 = \lambda$ for comparison with the one failure expressions. So that $\lim_{\lambda_2 \to \infty} E[A_2(t)] = E[A_1(t)]$.

In the two failure case set $\lambda_1 = \lambda$ and let $\lambda_2 \to \infty$ to obtain:

$$\lim_{\lambda_2 \to \infty} E[A_2^2(t)] = 4c^2\left[1 - e^{-\lambda t}\right] - 4bc\lambda\left[\frac{1}{\lambda^2} - \frac{t}{\lambda} e^{-\lambda t} - \frac{e^{-\lambda t}}{\lambda^2}\right]$$

$$+ b^2\lambda\left[-\frac{t^2}{\lambda} e^{-\lambda t} + \frac{2}{\lambda^3} - 2\left(t + \frac{1}{\lambda}\right)\frac{e^{-\lambda t}}{\lambda^2}\right]$$

For the one failure case the expression (with c replaced by $2c$) is:

$$E[A_1^2(t)] = \left(2c - \frac{b}{\lambda}\right)^2 + \frac{b^2}{\lambda^2} - \left[\left(2c - bt - \frac{b}{\lambda}\right)^2 + \frac{b^2}{\lambda^2}\right] e^{-\lambda t}$$

$$E[A_1^2(t)] = 4c^2 - \frac{4bc}{\lambda} + \frac{2b^2}{\lambda^2} - \left[4c^2 - 4bc\left(t + \frac{1}{\lambda}\right) + b^2\left(t + \frac{1}{\lambda}\right)^2 + \frac{b^2}{\lambda^2}\right] e^{-\lambda t}.$$

In this form direct comparison of the c^2, bc, and b^2 terms show that:

$$\lim_{\lambda_2 \to \infty} E[A_2^2(t)] = E[A_1^2(t)].$$

It follows from these results that $\lim_{\lambda_2 \to \infty} Var[A_2(t)] = Var[A_1(t)]$ as desired.

Thus the limit connection between the one failure and two failure cases is established.

3.5 Discussion

The expressions developed in this chapter represent a basic mathematical methodology for the study of incentive warranties when there are at most two failures. The results are complicated functions of the contract parameters, failure rates, and time. They can best be studied in detail by numerical evaluation and graphing of the representational quantities $E[A_2(t)]$, and $Var[A_2(t)]$. However special studies can be made without detailed numerical calculation. Such studies can address some specific questions of interest and provide insight about the model. A typical special consideration is given in this section, both for its intrinsic interest and as an example of the kind of study that can be made. Some considerations about the present value of money in the more than one failure case are also discussed. Some additional checks on complicated expressions using dimensional analysis are discussed at the end of this section.

In Section 3.3.2 an expression is given for $E_t[T_2]$, the expected value of the time to second failure up to time t when $\lambda_1 \neq \lambda_2$. If this expression is considered for the limit case with $\lambda_2 = \lambda$ and $\lambda_1 \to \lambda$ one obtains:

$$E_t[T_2] = \frac{2}{\lambda} + t \lim_{\lambda_1 \to \lambda} \frac{\lambda_1 e^{-\lambda t} - \lambda e^{-\lambda_1 t}}{\lambda - \lambda_1} + \lim_{\lambda_1 \to \lambda} \frac{\lambda_1^2 e^{-\lambda t} - \lambda^2 e^{-\lambda_1 t}}{\lambda_1^2(\lambda - \lambda_1)}$$

$$E_t[T_2] = \frac{2}{\lambda} - \left[2t + \lambda t^2 + \frac{2}{\lambda}\right] e^{-\lambda t} .$$

A special case of some interest is obtained when the expected time to failure is $T^*/2$ so that in a purely expected value sense the first failure occurs half way through the warranty period and the second failure occurs at the end. This can be thought of somewhat like the "one hoss shay" case of Chapter 2, with of course some major differences due to the exponential nature of the failure process. For this case $\lambda = 2/T^*$ and the expected payment by the end

of the warranty period is obtained by setting $t = T^*$ in the appropriate expressions. For this purpose one could use the quantity $E[A_2(t)]$ with $\lambda_1 = \lambda_2 = \lambda$. However the same result can be obtained by the following alternative procedure:

$$E[A_2(T^*)] = 2c\, P[T_2 \leq T^*] - b\, E_{T^*}[T_2] \ ,$$

since $P[T_2 \leq T^*] = 1 - 3\, e^{-\lambda T^*}$, and $E_{T^*}[T_2] = T^* - 5\, T^* e^{-\lambda T^*}$, the result becomes $E[A_2(T^*)] = 2c - b\, T^* + [5\, T^* - 6c]\, e^{-2}$.

When the second failure occurs at T^* the payment is equal to $2c - b\, T^*$ no matter when the first failure occurs. Thus if there is a deterministic second failure the total payment will be exactly $2c - b\, T^*$. The expected payment when $\lambda_1 = \lambda_2 = 2/T^*$ will be greater or less than the deterministic amount, depending on whether $5\, b\, T^* - 6c$ is positive or negative respectively. This observation provides an opportunity to relate the contract parameters b, c, and T^*. However the relation only holds for the special failure profile governed by the common failure rate $2/T^*$.

By having the second failure occur at T^*, given that two failures must occur in the warranty period, the contractor achieves maximum benefit from the fee pool. Therefore he will expect to pay no less than the amount $2c - b\, T^*$. Hence one obtains $5\, b\, T^* - 6c \geq 0$. It is a basic assumption of the present model methodology that the contractor can not profit from the failure events, imposing the condition $2c - b\, T^* \geq 0$. These two conditions lead to a rather tight relation between the three contract parameters, of the form:

$$2c \geq b\, T^* \geq 1.2c \ .$$

If the contractor is willing to have the expected payment less than $2c - b\, T^*$

he will have to assume a greater risk of having to pay more than the expected value. This is a less conservative position than the one described above. It imposes the condition: $5bT^* - 6c \leq 0$ which reduces to: $bT^* \leq 1.2c$ which is always consistant with the nonprofit assumption $bT^* \leq 2c$.

This kind of analysis can serve as a guide for the selection of contract parameters, with proper caution for the underlying assumptions about the random failure profile. The cost c and warranty period T^* are relatively easy to establish so that a condition such as $2c > bT^* \geq 1.2c$ provides a means of determining the incentive pool slope b by satisfying:

$$\frac{2c}{T^*} \geq b \geq \frac{1.2c}{T^*} .$$

When the present value of money is considered a number of questions arise. For example if the first failure event occurs at a time for which $c - bT_1 < 0$ there is an excess pool value which might be invested in such a way as to provide more future value than simply retaining it as part of the pool (the procedure used in the models of this chapter). Such excess pool investment would have to be allowed or disallowed by the warranty contract. One consideration in such a situation would be the possibility of allowing the excess as an award fee in the event of no future failure, or of requiring it to be used only when and if a second failure occurs. These are interesting possibilities for motivational contracting in these situations. Another possibility is for the contract to allow the contractor an option as to how much of the incentive fee pool he uses toward payment of the first failure. In a discounting (inflationary) environment one can argue that the full amount available should be spent at the failure time since it will only be worth less at any future expenditure. However if the contractor is allowed to invest a part of the fee

pool that is due to him at the failure time then he can balance investment income against the discount rate effect, and by assuming various levels of risk, hope to improve his position for payment of the second failure if it occurs. When no second failure occurs the contract must specify the disposition of such funds. Still another possibility is to allow the contractor to borrow against future levels of the fee pool. This is one way to overcome discounting effects by spending money at an earlier time. There are problems with this concept such as how to assess the contractor in the case of no second failure. A possibility seems to arise which may provide a negative motivation in which the contractor will actually desire a second failure in order to release the fee pool funds. More detailed investigation of value of money concepts is beyond the scope of this report. However they may be expected to play an important role in actual warranty contracts and lead to complications requiring careful analysis by both contractor and customer.

The expressions developed in this chapter are complicated and it is important to check them as thoroughly as possible against computational or theoretical error. A major class of such checks is given in Section 3.4. An additional type of check is to consider the dimension of each term in an expression to be sure they all represent the same quantity. This kind of dimensional analysis check is illustrated below for $E[A_2(t)]$ in the equal failure rate case. The quantity to be checked is:

$$E[A_2(t)] = 2c - \frac{2b}{\lambda} + \left[2b\left(\frac{1}{\lambda} + t + \frac{\lambda t^2}{2}\right) - 2c(1 + \lambda t)\right]e^{-\lambda t}$$

The dimensions of the various quantities are given in the following table where τ denotes "time" and γ denotes dollars:

Quantity	c	b	λ	t
Dimension	γ	γ/τ	$1/\tau$	τ

Since λt is dimensionless the factor $e^{-\lambda t}$ is dimensionally correct. The expression is now put into "dimensional" terms:

$$\text{dimension} \quad E[A_2(t)] = \gamma - \frac{\gamma\tau}{\tau} + \left[\frac{\gamma}{\tau}\left(\tau + \tau + \frac{1}{\tau}\tau^2\right) - \gamma\left(1 + \frac{1}{\tau}\tau\right)\right] .$$

It is seen that every term has the dimension of dollars as should be the case. This kind of check is particularly useful in finding mistakes such as missing factors or incorrect powers of variables.

Chapter 4. General Availability Case

Section 4.1 General Methodological Model

4.1.1 Model Form

In this chapter the payment process $A_k(t)$ for k failures is considered together with the general payment process $A(t)$ which does not depend on any specific assumption about the number of failures. It is important to observe that the payment processes are considered up to a time t which is of most interest when $t \leq T^*$, with T^* as the warranty period. The expression of results as a function of time t gives considerable structure to the availability warranty models and is imposed, in any case, by the finite warranty period T^*. However such a model representation causes restrictions in both the results and analysis techniques normally available for the study of failure processes. In the usual case there is no restriction on the time and full distributions of failure times can be employed. This is not the case in the present study as already observed in previous chapters for the one and two failure situations. The effect of the more complicated model structure imposed by the finite time condition is even more evident in the general considerations of this chapter.

The general methodological model for incentive availability warranties utilizes the following quantities:

$F_{T_k}(t) = P\left[T_k \leq t\right]$, the distribution function of the k^{th}, failure, denoted by the random variable T_k.

$f_{T_k}(v)$ = the density function for T_k

$E_t\left[T_k\right]$ = the partial expectation of T_k restricted to the range $T_k \leq t$:

$$E_t\left[T_k\right] = \int_0^t v\, f_{T_k}(v)dv$$

$E\left[A_k(t)\right]$ = expected payment under the warranty, by time $t \leq T^*$, when there are at most k failures.

$A_k(t)$ the payment process up to time t when there are at most k failures.

One could of course also define the variance of the payment process, $Var\left[A_k(t)\right]$, but for $k > 2$ this quantity *is an extremely complicated* function of t and the contract parameters. Therefore in the general case it is not employed as a representational feature of the model.

A(t) is the payment process up to time t no matter how many failures may occur.

E[A(t)] is the expected value of A(t) up *to time* t

N = the number of failures that occur

P[N = k] is the probability function for the discrete random variable N.

In the general methodology the random variable N is defined independently from the failure time random variables T_k. Of course the failure process imposes a relation between the failure times and the number of failures. For example the well known generation of exponential failure times by a Poisson failure process. However in the present methodology the point of view is different. As a contract negotiation decision position the parties to the contract may assume some a priori probability function P[N = k]. The generation of failures by the failure time process is unrelated to such an a priori specification of number of failures for the warranty period (or indeed for any time value t). In fact the generation process and the specification of failures may be related in various ways that involve considerations outside the scope of the

esearch reported on in this paper but falling within the considerations of future research as mentioned in Chapter 5.

The above definitions and considerations lead directly to the formulation of the general model in the following form:

$$E[A(t)] = \sum_{k=1}^{\infty} E\left[A_k(t)\right] P[N = k] ,$$

$$E\left[A_k(t)\right] = k\,c\,P\left[T_k \leq t\right] - b\,E_t\left[T_k\right] ,$$

$$P\left[T_k \leq t\right] = \int_0^t f_{T_k}(v)\,dv .$$

In these models $E[A(t)]$ and $E\left[A_k(t)\right]$ are partial expectations taken over the range $[0, t]$, generalizing similar results from Chapters 2 and 3.

If R_i is the random time from the i-1 st. to the i^{th} failure: $T_k = R_1 + R_2 + \ldots + R_k$ and $f_{T_k}(v)$ can be generated from the individual failure time distributions $f_{R_i}(v)$. In this report the only specific failure time random variables considered are exponential.

4.1.2 Approaches to Analysis

Because T_k is the sum of k independent random variables one might consider the possibility of computing the moments for T_k (or indeed its distribution form) by using characteristic (or moment) generating functions. Equivalently sums of expectation formulas suggest themselves. However these classical approaches, so useful in the usual full expectation situation become so involved in the partial expectation situation as to be of little value. Therefore it is necessary to proceed directly to the calculation of desired quantities. Before proceeding to consideration of such calculations some results will be

given for partial expectations to illustrate the problems they present for analysis and to record their form as a by-product of the present research. It should be explicitly recognized that there is nothing technically wrong with using the proper form of partial expectation quantities. The problem is not theoretical but practical application of the complicated forms involved.

As the first example of partial expectation calculations consider the random variable T_2, time to second failure. Since $T_2 = R_1 + R_2$ full expectation follows the well known formula $E\left[T_2\right] = E\left[R_1\right] + E\left[R_2\right]$ in which the marginal distributions for R_1 and R_2 can be used to compute $E\left[R_1\right]$ and $E\left[R_2\right]$. In terms of partial expectations a formal similarity exists as may be seen from the following development:

$$E_t\left[T_2\right] = \int_0^t \int_0^{t-r_2} (r_1 + r_2)\, f_{R_1, R_2}(r_1, r_2)\, dr_1\, dr_2$$

$$= \int_0^t \int_0^{t-r_2} r_1\, f_{R_1, R_2}(r_1, r_2)\, dr_1\, dr_2 + \int_0^t \int_0^{t-r_2} r_2\, f_{R_1, R_2}(r_1, r_2)\, dr_1\, dr_2$$

$$= E_t\left[R_1\right] + E_t\left[R_2\right] .$$

The derivation of this formal result clearly indicates the need to compute $E_t\left[R_i\right]$ in the multivariate sense due to the partial expectation effect. This considerably complicates the use of these formulas.

Though the use of partial expectations complicates calculations they do give correct results when properly employed. As an illustration in the exponential failure case with R_1 and R_2 independent random variables having failure rate parameters λ_1 and λ_2 respectively, direct calculation yields:

$$E_t\left[R_1\right] = \frac{1}{\lambda_1} - \frac{\lambda_1 e^{-\lambda_2 t}}{(\lambda_2 - \lambda_1)^2} + \left[\frac{2\lambda_1\lambda_2 - \lambda_2^2}{\lambda_1(\lambda_2 - \lambda_1)^2} - \frac{\lambda_2 t}{\lambda_2 - \lambda_1}\right] e^{-\lambda_1 t}$$

$$E_t\left[R_2\right] = \frac{1}{\lambda_2} - \frac{\lambda_2 e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)^2} + \left[\frac{2\lambda_1\lambda_2 - \lambda_1^2}{\lambda_2(\lambda_2 - \lambda_1)^2} - \frac{\lambda_1 t}{\lambda_1 - \lambda_2}\right] e^{-\lambda_2 t}$$

Then the formal expression $E_t\left[T_2\right] = E_t\left[R_1\right] + E_t\left[R_2\right]$ yields:

$$E_t\left[T_2\right] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \left[\frac{2\lambda_1\lambda_2 - \lambda_1^2}{\lambda_2(\lambda_2 - \lambda_1)^2} - \frac{\lambda_1 t}{\lambda_1 - \lambda_2} - \frac{\lambda_1}{(\lambda_2 - \lambda_1)^2}\right] e^{-\lambda_2 t}$$

$$+ \left[\frac{2\lambda_1\lambda_2 - \lambda_2^2}{\lambda_1(\lambda_2 - \lambda_1)^2} - \frac{\lambda_2 t}{\lambda_2 - \lambda_1} - \frac{\lambda_2}{(\lambda_2 - \lambda_1)^2}\right] e^{-\lambda_1 t}$$

In Section 3.3.2 the following result is given:

$$E_t\left[T_2\right] = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left[\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2} + t\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right) + \frac{e^{-\lambda_2 t}}{\lambda_2^2} - \frac{e^{-\lambda_1 t}}{\lambda_1^2}\right]$$

A little algebra shows that these two expressions are the same. The example illustrates the fact that the classical sum of expectation procedures are not particularly helpful in dealing with partial expectations.

Because of the increasing complexity of the calculations for more than two failures it would be desirable to use characteristic function methods to compute variance. However because of the partial expectations these methods, so powerful in classical application, do not help in the present application. This may be observed from the following three considerations.

1. The usual product formula $\varphi_{T_k}(z) = \prod_{j=1}^{k} \varphi_{R_i}(z)$ for independent random variables R_j does not hold at all for the partial characteristic function

defined as $\varphi_X(z,t) = E_t\left[e^{izX}\right]$ for any random variable X.

2. The composite partial characteristic functions $\varphi_{R_j}(z,t)$ are complicated to compute since they must be evaluated in the full multivariate environment, i.e.

$$\varphi_{R_j}(z,t) = \int\int\cdots\int_{r_1+r_2+\ldots+r_k\leq t} \underbrace{\prod_{n=1}^{k} \lambda_n e^{-\lambda_n r_n} d r_n}_{n\neq j} \lambda_j e^{-(\lambda_j - iz)r_j} d r_j$$

e.g. when k=3

$$\varphi_{R_1}(z,t) = \int_0^t \int_0^{t-r_1} \int_0^{t-r_1-r_2} \lambda_2\lambda_3\, e^{-\lambda_2 r_2} e^{-\lambda_3 r_3} dr_2\, dr_3\, \lambda_1\, e^{-(\lambda_1 - iz)r_1} dr_1,$$

which even for this relatively simple case (k=3) is not at all simple to evaluate.

3. The useful formulas for expectation and most particularly for variance no longer hold in the partial expectation situation.

$$\varphi_T(z,t) = E_t\left[e^{izT}\right] = \int_0^t e^{izv} f_T(v)\, dv$$

$$= P[T < t] + E_t[T]\, iz + E_t\left[T^2\right]\frac{(iz)^2}{2!} + \cdots$$

so that $\varphi_T(0,t) = P[T \leq t]$ unlike the classical characteristic function for which $\varphi_X(0) = 1$ greatly simplifying calculations.

The characteristic function may be used to compute the variance directly by the classical result:

$$\left.\frac{d^2 \log \varphi_X(z)}{dz^2}\right|_{z=0} = -\operatorname{Var}[X].$$

For partial expectations this result is greatly complicated. Direct calculation shows:

$$\mathrm{Var}_t[T] = \left(\frac{1-P}{P}\right)\left(E_t[T]\right)^2 - P\frac{d^2 \log \varphi_T(z,t)}{dz^2}\bigg|_{z=0}$$

where $iE_t[T] = P\dfrac{d \log \varphi_T(z,t)}{dz}\bigg|_{z=0}$

and $P = P[T \le t]$.

The above considerations of partial expectations show that the most reasonable approach to analysis of the general failure case is by means of direct calculation on the random variables T_k. Because of the complexity involved evaluation of $\mathrm{Var}_t\left[T_k\right]$ and hence $\mathrm{Var}\left[A_k(t)\right]$ must be omitted. Though it is feasible to compute $\mathrm{Var}_t\left[T_k\right]$ for some specific cases, such as k=3, the results would be extremely cumbersome. This is true even for the equal failure rates case which, though simpler than the general case, is still complicated as shown in Section 4.3.

The analysis will form the density function $f_{T_k}(v)$ for the random variables T_k and use it to produce the required quantities $P\left[T_k \le t\right]$, $E_t\left[T_k\right]$, and $E\left[A_k(t)\right]$. These may then be combined with the a priori values $P[N=k]$ to form $E[A(t)]$ the unconditional warranty payment process.

Section 4.2 Exponential Model With Unequal Failure Rates

In this section the times between failures R_i will be independent exponentially distributed random variables with failure rates λ_i. The time to the k^{th} failure is given by T_k where $T_k = R_1 + R_2 + \ldots + R_k$. To start the analysis the random variable T_3 is considered. Though the calculations involved are elementary in form they are rather complicated in execution and only the major steps will be given.

$$F_{T_3}(t) = P\left[T_3 \le t\right] = \int_0^t \int_0^{t-r_1} \int_0^{t-r_1-r_2} \lambda_1\lambda_2\lambda_3\, e^{-\lambda_3 r_3 - \lambda_2 r_2 - \lambda_1 r_1}\, dr_3\, dr_2\, dr_1$$

Direct calculation and some algebraic arrangement yields:

$$F_{T_3}(t) = 1 - e^{-\lambda_1 t} + \frac{\lambda_1 \lambda_3}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) + \frac{\lambda_1 \lambda_2}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}\left(e^{-\lambda_1 t} - e^{-\lambda_3 t}\right)$$

This distribution function for T_3 has the proper limit of one as t becomes unbounded. The density function for T_3 is obtained as the derivative of $F_{T_3}(t)$. It can be expressed in the form:

$$f_{T_3}(t) = \lambda_1\lambda_2\lambda_3\left[\frac{e^{-\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{e^{-\lambda_2 t}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{e^{-\lambda_3 t}}{\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right] .$$

As a check on this result it may be observed that:

$$\lim_{\lambda_3 \to \infty} f_{T_3}(t) = \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 (t)}\right) = f_{T_2}(t) .$$

Next the general form for $f_{T_k}(t)$ will be developed. It may be observed that:

$$P\left[T_k \leq t\right] = P\left[T_{k-1} + R_k \leq t\right] = \int_0^t \int_0^{t-w} f_{T_{k-1}}(w)\, \lambda_k\, e^{-\lambda_k r}\, dr\, dw$$

At this point it is helpful to introduce the simplifying notation $f_{T_k}(t) \equiv f_k(t)$. Then upon partial integration the equation becomes:

$$F_{T_k}(t) = \int_0^t f_{k-1}(w)\left[1 - e^{-\lambda_k (t-w)}\right] dw.$$

Applying Leibniz's rule to differentiate the integral expression yields:

$$f_k(t) = \int_0^t f_{k-1}(w)\, \lambda_k\, e^{-\lambda_k (t-w)}\, dw$$

which determines the densities $f_k(t)$ as solutions to an integral difference

equation. The solution is obtained by the guess and verify technique. Thus it may be verified by insertion into the above equation that the solution is given by the following expression:*

$$f_k(t) = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k} \frac{e^{-\lambda_i t}}{(\lambda_1-\lambda_i)(\lambda_2-\lambda_i)\cdots(\lambda_k-\lambda_i)}$$

in which the denominator products contain k-1 factors, with the zero factor $\lambda_i-\lambda_i$ absent. This result generalizes the quantity $f_3(t)$ previously obtained directly as the density function for T_3.

The quantities of interest for the incentive warranty model may now be calculated using the density function $f_k(t)$. Using the definitions of the quantities involved and simple integration yields:

$$P\left[T_k \le t\right] = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k} \frac{\left(1 - e^{-\lambda_i t}\right)}{\lambda_i(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} ,$$

$$E_t\left[T_k\right] = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k} \frac{1 - e^{-\lambda_i t} - t\lambda_i e^{-\lambda_i t}}{\lambda_i^2(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} .$$

Since there are k failures the repair/replacement cost is kc so that:

$$E\left[A_k(t)\right] = kc\,P\left[T_k \le t\right] - b\,E_t\left[T_k\right]$$

$$E\left[A_k(t)\right] = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k} \frac{1}{\lambda_i(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} \left[\left(kc - \frac{b}{\lambda_i}\right)\left(1 - e^{-\lambda_i t}\right) + bte^{-\lambda_i t}\right].$$

This result yields the previous results for $E\left[A_1(t)\right]$ and $E\left[A_2(t)\right]$ from Chapters 2 and 3 respectively. One value of the previous alternative derivations is that they provide checks on the general result presented here.

*See complement to Chapter 4.

Section 4.3 Exponential Model With Equal Failure Rates

In the context of warranty modeling, as in many other aspects of availability, the special case of equal failure rates for each failure is likely to be the most realistic. It is, of course, also simpler in form than the general failure rate model of Section 4.2. Intermediate situations in which some, but not all, failure rates are different are not considered in this report.

When $T_k = R_1 + \ldots + R_k$ and R_i are independent, exponential random variables with the same failure rate λ it is well known that the density function for T_k is a gamma density. A notation specifying the equal failure rate case is obtained by placing a diacritical mark over the function symbol.

Even though the density function is known to be gamma it may be obtained from the integral difference equation in the same way the general density was obtained in Section 4.2. From results in Chapters 2 and 3 the values for T_1 and T_2 are already available as:

$$\hat{f}_1(v) = \lambda\, e^{-\lambda v}$$

$$\hat{f}_2(v) = \lambda^2\, v e^{-\lambda v} \quad .$$

The integral difference equation yields:

$$\hat{f}_3(v) = \lambda^3 \frac{v^2}{2!} e^{-\lambda v} \, .$$

The result for arbitrary k, which may be shown to satisfy the integral difference equation is:

$$\hat{f}_k(v) = \frac{\lambda^k v^{k-1}}{(k-1)!} e^{-\lambda v} \, .$$

In the equal failure rates model a major role is played by the incomplete gamma function $\gamma(\alpha, x)$ defined as:

$$\gamma(\alpha, x) = \int_0^x e^{-v}\, v^{\alpha-1}\, dv \,.$$

The quantities required for the model may now be developed.

$$\hat{P}\left[T_k \leq t\right] = \int_0^t \frac{\lambda^k\, v^{k-1}\, e^{-\lambda v}\, dv}{(k-1)!} = \frac{1}{(k-1)!}\, \gamma(k, \lambda t) \;.$$

$$\hat{E}_t\left[T_k\right] = \int_0^t \frac{\lambda^k\, w^k}{(k-1)!}\, e^{-\lambda w} dw = \frac{k}{\lambda}\, \hat{P}\left[T_{k+1} < t\right]$$

$$\hat{E}_t\left[T_k\right] = \frac{1}{\lambda(k-1)!}\, \gamma(k+1, \lambda t)$$

$$\hat{E}\left[A_k(t)\right] = k\,c\, \hat{P}\left[T_k \leq t\right] - b\, \hat{E}_t\left[T_k\right]$$

$$= \frac{1}{(k-1)!}\left[k\,c\, \gamma(k, \lambda t) - \frac{b}{\lambda}\, \gamma(k+1, \lambda t)\right] .$$

The recurrence relation:

$$\gamma(k+1, x) = k\, \gamma(k, x) - x^k\, e^{-x} \,,$$

may be used to obtain the following alternative form for $\hat{E}\left[A_k(t)\right]$:

$$\hat{E}\left[A_k(t)\right] = \frac{1}{(k-1)!}\left[k\left(c - \frac{b}{\lambda}\right) \gamma(k, \lambda t) + b\, \lambda^{k-1}\, t^k\, e^{-\lambda t}\right].$$

The unconditional payment process has expectation of the form:

$$\hat{E}\left[A(t)\right] = \sum_{k=1}^{\infty} E\left[A_k(t)\right]\, P[N=k]$$

$$= \left(c - \frac{b}{\lambda}\right) \sum_{k=1}^{\infty} \frac{k}{(k-1)!}\, \gamma(k, \lambda t)\, P[N=k] + \frac{b\, e^{-\lambda t}}{\lambda} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}\, P[N=k]$$

The a priori probabilities P[N=k] can be selected to represent any particular kind of failure model one might wish to study. They should indicate

what is perceived to be the situation in whatever actual availability profile is being studied. One likely (though not necessary) property of $P[N=k]$ might be that they should decrease with increasing k so that one failure is most likely, two failures less likely, and so forth. Since the consideration is for a finite time, the warranty period, such a priori considerations seem reasonable. A specific form for such a priori probabilities is: $P[N=k] = (e-1)\, e^{-k}$. Using this particular example gives:

$$\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} P[N=k] = \left(1-e^{-1}\right)\lambda t \exp(\lambda t/e)$$

so that, for this particular example:

$$\hat{E}[A(t)] = \left(c - \frac{b}{\lambda}\right)(e-1) \sum_{k=1}^{\infty} \frac{k\, e^{-k}}{(k-1)!} \gamma(k, \lambda t) + b\left(1-e^{-1}\right) t\, e^{-\lambda t} \exp(\lambda t/e)$$

The results obtained above can be given in an alternative form using the relation:

$$\gamma(k, x) = (k-1)! \left[1 - e^{-x} e_{k-1}(x)\right]$$

where $e_{k-1}(x)$ is the truncated exponential series defined as:

$$e_{k-1}(x) = \sum_{m=0}^{k-1} \frac{x^m}{m!} .$$

The expected payment process for general $P[N=k]$ has the form:

$$\hat{E}[A(t)] = \left(c - \frac{b}{\lambda}\right) \sum_{k=1}^{\infty} k\left[1 - e^{-\lambda t} e_{k-1}(\lambda t)\right] P[N=k]$$

$$+ \frac{b\, e^{-\lambda t}}{\lambda} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} P[N=k]$$

$$\hat{E}[A(t)] = \left(C - \frac{b}{\lambda}\right) E[N] + e^{-\lambda t} \sum_{k=1}^{\infty} \left[\frac{b(\lambda t)^k}{\lambda(k-1)!} - k\left(c - \frac{b}{\lambda}\right) e_{k-1}(\lambda t)\right] P[N=k] \ .$$

Section 4.4 Limit Results and Checks

Three kinds of limit situations may be considered for the incentive warranty models presented in this chapter. Such limits provide checks on the analysis and in some cases give useful bounding forms for the results. They also allow a comparison with intuitive impressions about the model behavior in special situations. One kind of limit is to reduce the time between the last two failures to zero. This was used as a check on calculations in Chapter 3 and serves the same purpose here. Another type of limit is to remove the finite time restriction so that full expectations can be used. Such limit results give simple forms that can be more easily appreciated from an intuitive viewpoint than the complicated, finite time results. A third type of limit is to allow all the failure rates to become the same in the general model to obtain the equal failure rates case. This seems to be a particularly difficult limit process and only partial results were obtained as given in Section 4.4.3.

4.4.1 Coalescence of Last Failures

As an example of the coalescence of last failures consider the density function $f_3(t)$ which should become $f_2(t)$ in the limit as $\lambda_3 \to \infty$. The calculation follows:

$$\lim_{\lambda_3 \to \infty} f_3(t) = \lambda_1 e^{-\lambda_1 t} + \left(\lim_{\lambda_3 \to \infty} \frac{\lambda_3}{\lambda_2 - \lambda_3}\right) \frac{\lambda_1}{\lambda_1 - \lambda_2} \left(\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}\right)$$

$$+ \lim_{\lambda_3 \to \infty} \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)} \left(\lambda_3 e^{-\lambda_3 t} - \lambda_1 e^{-\lambda_1 t}\right)$$

The last limit is zero and the other limit is -1 so that result is:

$$\lim_{\lambda_3\to\infty} f_3(t) = \left(\lambda_1 - \frac{\lambda_1^2}{\lambda_1-\lambda_2}\right) e^{-\lambda_1 t} + \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2} e^{-\lambda_2 t}$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) = f_2(t) \ .$$

The general coalescence result can be established from the integral difference equation:

$$f_k(t) = \int_0^t f_{k-1}(w)\, \lambda_k\, e^{-\lambda_k(t-w)} dw$$

$$= \lambda_k e^{-\lambda_k t}\left[f_{k-1}(t)\frac{e^{\lambda_k t}}{\lambda_k} - \frac{f_{k-1}(0)}{\lambda_k} - \int_0^t f'_{k-1}(w)\frac{e^{\lambda_k w}}{\lambda_k} dw\right]$$

$$\lim_{\lambda_k\to\infty} f_k(t) = f_{k-1}(t) + \lim_{\lambda_k\to\infty}\left[-e^{-\lambda_k t} f_{k-1}(0) - \int_0^t f'_{k-1}(w)\, e^{-\lambda_k(t-w)} dw\right] \ .$$

From which the general result follows:

$$\lim_{\lambda_k\to\infty} f_k(t) = f_{k-1}(t) \ .$$

The same argument applies to the equal failure rate case in the sense that one λ is allowed to become unbounded while all the others remain fixed in value.

4.4.2 Unbounded Failure Time

The quantity $P[T_k \le t]$ should of course approach one as t becomes unbounded. It is the distribution function $F_{T_k}(t)$. For the equal failure rate case the form of $\hat{F}_{T_k}$ shows at once that $\lim_{t\to\infty} \hat{F}_{T_k}(t) = 1$. This result is well

known as a consequence of the fact that T_k has a gamma distribution in the equal failure rate case.

The unequal failure rate expression is:

$$\lim_{t\to\infty} F_{T_k}(t) = \lambda_1 \ldots \lambda_k \sum_{i=1}^{k} \frac{1}{\lambda_i(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} = A_k,$$

and it is by no means obvious that this expression is equal to one. Direct algebraic procedures show that for $k=1,2,3$ the quantity A_k is indeed one. However such procedures become extremely involved for larger values of k. One approach is to show that the expression A_k is independent of the λ_j quantities and then evaluate it for a particular set of values to show that it is the constant one. However it seems to be difficult to show for example that the derivative with respect to a particular λ_j is zero, from which the constant nature of the quantity A_k would follow.

Another approach can be taken which shows that A_k is constant. The quantity A_k can be shown to be constant by using the integral difference equation from page 4.8. By direct verification it has been established that

$f_k(t) = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k} \frac{e^{-\lambda_i t}}{(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)}$ satisfies the equation. If this form is used to express $f_{k-1}(w)$ in the integrand of the equation, upon integration $f_k(t)$ must result. The calculation of the integral follows:

$$\int_0^t \left[\lambda_1 \cdots \lambda_{k-1} \sum_{i=1}^{k-1} \frac{\lambda_k e^{-\lambda_i w} e^{-\lambda_k(t-w)}}{(\lambda_1-\lambda_i)\cdots(\lambda_{k-1}-\lambda_i)}\right] dw$$

$$= \lambda_1 \cdots \lambda_{k-1}\lambda_k \left[\sum_{i=1}^{k-1} \frac{e^{-\lambda_i t}}{(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} - e^{-\lambda_k t} \sum_{i=1}^{k-1} \frac{1}{(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)}\right]$$

This must equal $f_k(t)$ which can be expanded as follows:

$$f_k(t) = \lambda_1 \cdots \lambda_k \left[\sum_{i=1}^{k-1} \frac{e^{-\lambda_i t}}{(\lambda_1 - \lambda_i) \cdots (\lambda_k - \lambda_i)} + \frac{e^{-\lambda_k t}}{(\lambda_1 - \lambda_k) \cdots (\lambda_{k-1} - \lambda_k)} \right] .$$

Setting the two expansions for $f_k(t)$ equal yields:

$$\sum_{i=1}^{k-1} \frac{1}{(\lambda_1 - \lambda_i) \cdots (\lambda_k - \lambda_i)} = \frac{-1}{(\lambda_1 - \lambda_k) \cdots (\lambda_{k-1} - \lambda_k)}$$

Now consider:

$$A_{k-1} = \lambda_1 \cdots \lambda_{k-1} \sum_{i=1}^{k-1} \frac{1}{\lambda_i(\lambda_1 - \lambda_i) \cdots (\lambda_{k-1} - \lambda_i)}$$

$$A_{k-1} = \lambda_1 \cdots \lambda_{k-1} \sum_{i=1}^{k-1} \frac{\lambda_k - \lambda_i}{\lambda_i(\lambda_1 - \lambda_i) \cdots (\lambda_k - \lambda_i)}$$

$$A_{k-1} = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k-1} \frac{1}{\lambda_i(\lambda_1 - \lambda_i) \cdots (\lambda_k - \lambda_i)}$$

$$- \lambda_1 \cdots \lambda_{k-1} \sum_{i=1}^{k-1} \frac{1}{(\lambda_1 - \lambda_i) \cdots (\lambda_k - \lambda_i)}$$

Using the result obtained above from the integral difference equation for the last summation yields:

$$A_{k-1} = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k-1} \frac{1}{\lambda_i(\lambda_1 - \lambda_i) \cdots (\lambda_k - \lambda_i)} + \frac{\lambda_1 \cdots \lambda_{k-1}}{(\lambda_1 - \lambda_k) \cdots (\lambda_{k-1} - \lambda_k)} \frac{\lambda_k}{\lambda_k}$$

Multiplication of numerator and denominator of the last term by λ_k as shown shows that:

$$A_{k-1} = \lambda_1 \cdots \lambda_k \sum_{i=1}^{k} \frac{1}{\lambda_i(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} = A_k$$

Thus A_k does not depend on k. It is a constant as required. The constant expression A_k can be evaluated by assigning the values $\lambda_i = i$ for $i = 1, \cdots, k$. The result is:

$$A_k = k! \sum_{i=1}^{k} \frac{1}{i(1-i)(2-i)\cdots(k-i)}$$

$$= k! \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i(i-1)(i-2)\cdots(1)\,(1)(2)\cdots(k-i)}$$

$$= k! \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i!\,(k-i)!} = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i}$$

Since $\sum_{i=0}^{k} \binom{k}{i}(-1)^i = 0$, it follows that $A_k - 1 = 0$ or $A_k = 1$ as required.

Another limit consideration for unbounded t is the study of the payment process A(t).

For the equal failure rate case it was found that $\hat{E}_t\left[T_k\right] = \frac{k}{\lambda}\,\hat{R}\left[T_{k+1} \le t\right]$ so that $\lim_{t\to\infty} \hat{E}_t\left[T_k\right] = \frac{k}{\lambda}$ since $\lim_{t\to\infty} \hat{P}\left[T_k \le t\right] = 1$ for all k. Therefore:

$$\lim_{t\to\infty} \hat{E}\left[A_k(t)\right] = kc - \frac{bk}{\lambda} \quad , \quad \text{and}$$

$$\lim_{t\to\infty} \hat{E}[A(t)] = \sum_{k=1}^{\infty} \lim_{t\to\infty} E\left[A_k(t)\right] P[N=k]$$

$$= \left(c - \frac{b}{\lambda}\right) \sum_{k=1}^{\infty} k\, P[N=k]$$

$$= \left(c - \frac{b}{\lambda}\right) E[N] \ .$$

This result is intuitively satisfying, indicating in the full expectation case that the payment under the warranty is expected to equal the product of the payment per failure times the expected number of failures.

Two special examples of this result are obtained from expressions derived in Section 4.3. One form of the expectation was found to be:

$$\hat{E}[A(t)] = \left(c - \frac{b}{\lambda}\right) E[N] + e^{-\lambda t} \sum_{k=1}^{\infty} \left[\frac{b(\lambda t)^k}{\lambda(k-1)!} - k\left(c - \frac{b}{\lambda}\right) e_{k-1}(\lambda t)\right] P[N=k]$$

and this yields the above limit as $t\to\infty$ from a different point of view $\Big($i.e. a detailed formula for $\hat{E}(t)\Big)$.

When $P[N=k] = (e-1)\, e^{-k}, k \geq 1$, it was found that:

$$\hat{E}[A(t)] = \left(c - \frac{b}{\lambda}\right)(e-1) \sum_{k=1}^{\infty} \frac{k\, e^{-k}}{(k-1)!} \gamma(k, \lambda t) + \frac{b}{e}(e-1) t\, e^{-\lambda t\left(1 - \frac{1}{e}\right)} \ .$$

Since $\lim_{t\to\infty} \gamma(k, \lambda t) = \Gamma(k) = (k-1)!$, and the last term goes to zero as t becomes unbounded, one obtains:

$$\lim_{t\to\infty} \hat{E}[A(t)] = \left(c - \frac{b}{\lambda}\right)(e-1) \sum_{k=1}^{\infty} k\, e^{-k} .$$

In this expression $E[N] = (e-1) \sum_{k=1}^{\infty} k\, e^{-k} = \frac{e}{e-1} \approx 1.6$. It may be noted that the value $E[N] = 1.6$ for this special a priori distribution of the number of failures indicates that it is a reasonable one to assume if one and two failures seem about equally likely and more failures seem unlikely.

4.4.3 Approaches to Equilization of Failure Rates

It would be interesting to show that the limit of general expressions for the unequal failure rates case tend to corresponding expressions in the equal failure rate case as the failure rates approach the same value. This seems to be a difficult limit process to evaluate and only a little actual progress has been made toward its study. Though several attempts at taking the limit seemed promising none succeeded in yielding results. The problem with the general analysis is the complex interaction of indeterminate forms. For their study it is necessary to deal with products containing many factors, differentiation of which is extremely involved. On the other hand algebraic procedures also become extremely complex. One result, forming a partial contribution to the analysis, is the following:

$$\lim_{\lambda_1, \lambda_2, \lambda_3 \to \lambda} f_3(t) = \hat{f}_3(t)$$

This result may be shown by direct calculations, starting from the expression for $f_3(t)$ in the form:

$$f_3(t) = \lambda_1\lambda_2\lambda_3 \left[\frac{e^{-\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{e^{-\lambda_3 t}}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} - \frac{e^{-\lambda_2 t}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} \right] .$$

Let $\lambda_1 = \lambda$ to obtain:

$$\lim_{\lambda_1\to\lambda} f_3(t) = \lambda\lambda_2\lambda_3\, e^{-\lambda t}\left[\frac{1}{(\lambda-\lambda_2)(\lambda-\lambda_3)} + \frac{e^{-(\lambda_3-\lambda)t}}{(\lambda_2-\lambda_3)(\lambda-\lambda_3)} - \frac{e^{-(\lambda_2-\lambda)t}}{(\lambda-\lambda_2)(\lambda_2-\lambda_3)}\right]$$

When $\lambda_2 \to \lambda$ it is necessary to evaluate the expression:

$$A = \lim_{\lambda_2\to\lambda}\left[\frac{1}{(\lambda-\lambda_2)(\lambda-\lambda_3)} - \frac{e^{-(\lambda_2-\lambda)t}}{(\lambda-\lambda_2)(\lambda_2-\lambda_3)}\right] ,$$

using some algebra and L'Hospital's rule yields:

$$A = -\frac{\left(1+(\lambda-\lambda_3)t\right)}{(\lambda-\lambda_3)^2} ,$$ so that

$$\lim_{\lambda_1,\lambda_2\to\lambda} f_3(t) = \lambda^2\lambda_3\, e^{-\lambda t}\left[\frac{e^{-(\lambda_3-\lambda)t}}{(\lambda-\lambda_3)^2} - \frac{1}{(\lambda-\lambda_3)^2} - \frac{t}{(\lambda-\lambda_3)}\right]$$

Similar treatment of this expression yields:

$$\lim_{\lambda_1,\lambda_2,\lambda_3\to\lambda} f_3(t) = \lambda^3\, e^{-\lambda t}\,\frac{t}{2}\lim_{\lambda_3\to\lambda}\frac{e^{-(\lambda_3-\lambda)t} - 1}{\lambda-\lambda_3}$$

and finally:

$$\lim_{\lambda_1,\lambda_2,\lambda_3\to\lambda} f_3(t) = \lambda^3\, e^{-\lambda t}\,\frac{t^2}{2} = \hat{f}_3(t)$$ as required.

One might hope that this kind of procedure could be extended to the general case. However attempts to do so have remained ineffective so far in the present research. Though interesting,such considerations are not part of the main analysis and are not necessary for the continued development of the study of incentive warranty models.

Complement to Chapter 4

The varification that $f_k(t)$ satisfies the integral difference equation may be carried out as follows: Let

$$R = \int_0^t f_{k-1}(w)\,\lambda_k\, e^{-\lambda_k(t-w)}\,dw \quad \text{where it is assumed that}$$

$$f_{k-1}(w) = \lambda_1 \cdots \lambda_{k-1} \sum_{i=1}^{k-1} \frac{e^{-\lambda_i w}}{(\lambda_1 - \lambda_i) \cdots (\lambda_{k-1} - \lambda_i)}$$

and $\lambda_k \neq \lambda_i$ for $i = 1, \cdots, k-1$.

Then by integration:

$$R = \lambda_1 \cdots \lambda_k \left[\sum_{i=1}^{k-1} \frac{e^{-\lambda_i t}}{(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} - e^{-\lambda_k t} \sum_{i-1}^{k-1} \frac{1}{(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} \right]$$

Thus it will be established that $R = f_k(t)$ if it can be shown that:

$$\sum_{i=1}^{k-1} \frac{1}{(\lambda_1-\lambda_i)\cdots(\lambda_k-\lambda_i)} = \frac{-1}{(\lambda_1-\lambda_k)\cdots(\lambda_{k-1}-\lambda_k)}$$

Consider the partial fraction expansion of the quantity on the right to yield an algebraic identity, valid for all values of λ_k (except of course, that $\lambda_k \neq \lambda_i$ for $i=1,\ldots,k-1$). The expansion is

$$\frac{-1}{(\lambda_1-\lambda_k)\cdots(\lambda_{k-1}-\lambda_k)} = \frac{B_1}{(\lambda_1-\lambda_k)} + \frac{B_2}{(\lambda_2-\lambda_k)} + \cdots + \frac{B_{k-1}}{(\lambda_{k-1}-\lambda_k)}$$

numerator of both sides must be equal so that:

$$-1 = B_1(\lambda_2-\lambda_k)\cdots(\lambda_{k-1}-\lambda_k) + B_2(\lambda_1-\lambda_k)\cdots(\lambda_{k-1}-\lambda_k)$$
$$+ \cdots + B_{k-1}(\lambda_1-\lambda_k)\cdots(\lambda_{k-2}-\lambda_k)$$

Let $\lambda_k = \lambda_1$ to obtain:

$$B_1 = \frac{-1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_{k-1}-\lambda_1)} .$$

Let $\lambda_k = \lambda_2$ to obtain:

$$B_2 = \frac{-1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\cdots(\lambda_{k-1}-\lambda_2)} .$$

And so forth, finally letting $\lambda_k = \lambda_{k-2}$ to obtain

$$B_{k-1} = \frac{-1}{(\lambda_1-\lambda_{k-1})\cdots(\lambda_{k-2}-\lambda_{k-1})} .$$

Putting in these values and interchanging the last factor in each denominator, thereby changing the sign, yields the desired summation. This establishes $f_k(t)$ as the solution form for the integral difference equation (page 4.8).

Chapter 5 Extensions

The study of incentivised availability warranties carried out as described in this report suggest a number of additional investigations. These range from technical and theoretical questions to implementation of the methodology for structuring warranty contracts. Various extensions of the methodological framework may also be identified. A number of these considerations are set forth in this chapter as a matter of record and to serve as possible guidelines for subsequent research.

Applicability

The very concept of incentive availability warranties implies application to actual contract structure either directly or in some modified form. Its analysis has no meaning except as related, in some context, to application. The present theoretical introduction of a methodology for study of these concepts may be carried toward application in several ways. The rather complex forms obtained for variance and expectation may be investigated in numerical (expressed as graphical) terms. The variance is a measure of risk assumption and as such plays a central role in sophisticated contract structuring. However the formulas for variance in the important one and two failure situations are complicated functions of the contract parameters, difficult to appreciate in terms of analytic expression. Numerical investigation of these formulas and of the expected values for both special and general situation formulas will carry the methodology further toward actual representational analysis of incentive warranty structures. A major goal of contract structure analysis is to study the interplay between motivation of contractor toward customer goals and the division of risk assumption in a potentially profitable environment. Further detailed study of the methodological formulas will contribute to this basic consideration.

Another feature related to application of the theory to actual warranty development is the interplay between the kinds of risks a contractor must assume. In a design decision context the contractor may expend money in the manufacturing phase to reduce the risk of spending money in the warranty period. Incentivisation of warranties would reduce the motivation for a contractor to so reduce warranty cost and hence work counter to high availability. On the other hand the incentivisation may motivate the contractor to agree to a warranty contract. Thus the situation is complex for both parties and would doubtless benefit from further investigation.

Applicability studies are therefore of two distinct types: investigation of how desirable the incentivising concept actually is in various contexts, and detailed, numerical study to provide contextual appreciation of the methodological results.

Extensions of Methodology

There are a number of considerations that would extend the potential application of the methodology by making it representational of a broader class of situations. Some of these deal with the incentive pool structure, some with the underlying stochastic environment, and some with the model structure itself. A number of these extensions are briefly identified below:

- Change in the form of the incentive fee pool may allow alternative interpretations and anlayses. As a simple illustration an exponential type fee pool may be used. In the single failure (reliability) case the payment process has the form $A_1(t) = c - a_o e^{\mu t}$. With exponential failure this model gives:

$$E\left[A_1(t)\right] = c\left(1 - e^{-\lambda t}\right) - \frac{a_o \lambda}{\lambda - \mu}\left(1 - e^{-(\lambda - \mu)t}\right)$$

dividing nicely into the payment without incentive pool and the contribution made by the incentive fee.

• The fee pool form can change after a failure rather than always return to the same rate of increase. The change can a fixed value, depending on the failure time, or be a random variable the value of which is specified by a probability distribution assumed as part of the model.

• Alternative contract forms can be used to produce motivation similar to the incentive contract approach. For example the award fee concept can be applied to warranty contracting. Awards can be designed to offset contractor payments under the warranty (as the incentive pool does). They can be based on the occurrence of failure events, the nature of the failure, or the time between observed failures.

• In the exponential failure model some of the failure rates can be the same and others different. This is a potentially useful model falling between the two extremes considered in Chapter 4. In particular it would be interesting to study one case, e.g. for k=3, in detail. Such a study may be expected to shed insight on the general case.

• The failure rates can be considered as variable, leading to more general failure distributions such as the Weibull.

• In the present study uniform failure time distributions were used only for the one failure case. However these distributions provide detailed instruments for some investigations of the payment process. Their use can be extended to more than one failure situations.

• The cost to repair/replace an item need not be constant. Variable service cost can be of different kinds, either being selected from a (small) number of discrete values or as a value of a continuous random variable. An interesting situation is when cost is represented by a mixed random variable in which a fixed replacement cost has non zero probability and a range of repair costs are continuously distributed. Consideration of replacement/repair time

and cost will increase the potential applicability of the methodology and indicate the effect of such factors on the concepts of incentivised warranties.

- The time to repair/replace can be explicitely considered in the methodology. The service time can be thought of as a single random variable (deterministic service time is a special case) or for more detailed study it can be represented as a sum of waiting time, diagnosis time, and actual repair (replacement) time. Its consideration alone can represent an extensive area for research and it is necessary to investigate only the general effect of service time on the availability profile as it relates to the payment process.

- The concepts of availability warranties can be extended to logistic warranties in which additional features of the logistic process are brought into the methodological considerations. An instance of such additions is the multiple equipment warranty in which multiple failures can occur and require service. The service queue enters into the considerations as does the full logistic pipeline. Analysis involving such ideas should be limited to considerations having the strongest impact on the payment process. It seems likely that features of the service queue (such as the number of service channels and the queue discipline) may be important in this context.

- Considerations related to the value of money are important in applying the incentive warranty methodology to actual contracts. These considerations are distinct from the availability theory approach to the model methodology but are associated with it in the formulation of realistic contracts. There are many distinct possibilities for dealing with incentive fee pool dollars and contractor payments from the discounting point of view.

Discounting can simply be incorporated into the model as a correction calculation on the value of money.

A more interesting situation is when the contractor can select less than

the available fee pool money and retain some toward the future failure expense. When the value of money is decreasing this seems like a bad idea since the full amount available will never be worth more to the contractor than when it becomes available. On the other hand the contractor may have an alternate use for the money that may increase its value in the future. It can be seen that there are many possibilities. To be used, they must be allowed under contract specification and their motivational effect on the contractor must be carefully studied. If conducted in detail such considerations become an additional dimension for the incentivised warranty methodology.

Technical Considerations

The result of allowing different failure rates to become equal is a particularly difficult limit evaluation that would contribute to the internal consistence of the research.

These things are not required for the main line of methodological development but do present some technical features of the mathematical analysis that could benefit from further study.

Theoretical Questions

The use of what has been called partial expectations in this report introduces questions about the role of this concept within the general framework of applied probability. It is not a conditional expectation though it is used to compute conditional expectations in the usual sense, e.g. $E\left[A_k(t)\right]$ a conditional expectation whereas $E_t\left[T_k\right]$ is not. Some study of this concept has been given in Section 4.1.2.

The other major theoretical consideration is the use of a priori probabilities for $P[N=k]$ and the relation between such probabilities and those generated by the failure time distributions. Because of the time boundary imposed by the warranty period the failure process does not develop under

the operation of the underlying stochastic process. The point of view is rather one in which the number of failures is specified and act to condition the process leading to the use of partial expectations of failure time and conditional expectations of payment up to a time t less than the warranty period. However it may be that the usual process for generating failures should play a role in the incentive availability warranty methodology.

In the general availability model the number of failures generated by the stochastic failure process, N_t, is a random variable of interest in addition to the failure time distribution. This random variable, generated by the process, is distinct from N the number of failures postulated for the equipment as an a priori feature of the model. The two variables N_t and T, one discrete and one continuous, complement each other as representiational factors of the random process. If T and N_t are the failure time and number of failures to time t respectively the relation between them is expressed by the equation $F_T(t) = 1 - P[N_t = 0]$. By differential difference equations the probability distribution for N_t can be determined from $F_T(t)$, the distribution function for T. In such relations some assumptions about the process must also be made (such as independence, or non-cascade assumptions). The study of the role of N_t contributes to the understanding of any stochastic process. This is true of the payment process model of incentive availability warranties, where there is the added feature of possible application of the model to actual contract structuring. Potential application is enhanced by being able to consider the model from different viewpoints.

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